Consider the linear system

\[ Dx = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad D = \frac{d}{dt}, \quad (1.1) \]

where \( A(t) \) is an \( n \times n \) matrix of real-valued continuous and bounded functions of the real variable \( t \) on the non-negative half-line. We say that

i) \( A(t) \) is a right Lappo-Danilevskii matrix \( A \in LD_r(s) \) if there exists \( s, s \geq 0 \), such that for all \( t \geq s \)

\[ A(t) \int_s^t A(u)du = \int_s^t A(u)du A(t); \quad (2) \]

ii) \( A(t) \) is a left Lappo-Danilevskii matrix \( A \in LD_l(s) \) if there exists \( s, s > 0 \), such that \( (2) \) is fulfilled for all \( 0 \leq t \leq s \);

iii) \( A(t) \) is a bilateral Lappo-Danilevskii matrix \( A \in LD_b(s) \) if there exists \( s, s \geq 0 \), such that \( (2) \) is fulfilled for all \( t \geq 0 \).

The corresponding systems \((1.1)\) are called right, left or bilateral Lappo-Danilevskii systems (cf. [1, p. 117]). In this paper we present some results on the distribution of the Lappo-Danilevskii systems among linear systems.

Let \( \rho(A, B) = \sup_{t \geq 0} \|A(t) - B(t)\| \), where \( \| \cdot \| \) be an arbitrary matrix norm, and let

\[ LD_r = \bigcup_{s \geq 0} LD_r(s), \quad LD_l = \bigcup_{s > 0} LD_l(s), \quad LD_b = \bigcup_{s \geq 0} LD_b(s). \]

Let, for simplicity, \( n = 2 \).

**Theorem 1.** Among linear differential systems there is a linear system \((1.1)\) such that for some \( \varepsilon > 0 \) the system \((1.1 + Q)\) is neither a bilateral nor a right Lappo-Danilevskii system for any matrix \( Q \) such that \( \rho(A, A + Q) \leq \varepsilon \).

**Theorem 2.** Among linear differential systems there is a linear system \((1.1)\) such that for any \( s > 0 \) there exists \( \varepsilon > 0 \) such that the matrix \( A + Q \notin LD_r(s) \) for any matrix \( Q \) such that \( \rho(A, A + Q) \leq \varepsilon \).

To prove these theorems it is sufficient to consider the matrix \( A(t) = (a_{ij}(t)), \ i, j = 1, 2, \) where \( a_{11}(t) = \sin(t + 1), \ a_{12}(t) = 1, \ a_{21}(t) = \exp(-t), \ a_{22}(t) = \cos(t + 1) \).

(Let the symbol \( [ \cdot, \cdot ] \) be used to indicate the Lie brackets, and let \( \cdot i, j \) be \((i, j)\)-element of the matrix \([ \cdot ] \). We have

\[ [A(t) + Q(t), \int_s^t A(u) + Q(u)du] = [A(t), \int_s^t A(u)du] + [A(t), \int_s^t Q(u)du] + \\
+[Q(t), \int_s^t A(u)du] + [Q(t), \int_s^t Q(u)du]. \]

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It is easy to verify that if $\rho(A, A+Q) \leq \varepsilon$, then for all $t \geq 0$, $s \geq 0$, and for all sufficiently small $\varepsilon$ we have:

$$
\left| A(t), \int_s^t Q(u)du \right|_{11} \leq 4\varepsilon|t-s|,
$$

$$
\left| A(t), \int_s^t Q(u)du \right|_{12} \leq 4\varepsilon|t-s|,
$$

$$
\left| Q(t), \int_s^t A(u)du \right|_{11} \leq 4\varepsilon|t-s|,
$$

$$
\left| Q(t), \int_s^t A(u)du \right|_{12} \leq 4\varepsilon|t-s|.
$$

Therefore, $\forall t \geq 0, s \geq 0$ we have

$$
F_{12}(t, s) = \left| A(t) + Q(t), \int_s^t (A(u) + Q(u))du \right|_{12} \geq \left| A(t), \int_s^t A(u)du \right|_{12} - 12\varepsilon|t-s|.
$$

Set $t_k = \exp((\pi/2 + 2k\pi) - 1, k \in \mathbb{N}$. It follows that $F_{12}(t_k, s) \geq \left| t_k - s - (s + 1)\cos\ln(s+1) - 12\varepsilon|t_k - s|$. It is easy to see that for sufficiently large $k$ we have $F_{12}(t_k, s) > 0$, so $F_{12}(t, s) \neq 0$. Thus $A + Q \notin LD_v$ and $A + Q \notin LD_v$

Similarly one can show that

$$
F_{11}(t, s) = \left| A(t) + Q(t), \int_s^t (A(u) + Q(u))du \right|_{11} \geq \left| \exp(-s) - \exp(-t) - (t - s)\exp(-t) - 12\varepsilon|t-s|.
$$

Since $F_{11}(0, s) \geq \left| \exp(-s) - 1 + s - 12\varepsilon$ and $\left| \exp(-s) - 1 + s\right| > 0$ for all $s > 0$, we see that $F_{11}(t, s) \neq 0$, i.e., $A + Q \notin LD_1(s)$.

**Theorem 3.** For any Lappo-Danilevskii system $(A_1)$ and for any $\varepsilon$ there exists a system $(B_1)$ such that $\rho(A_1, B_1) \leq \varepsilon$ but $(B_1)$ is not a Lappo-Danilevskii system.

Indeed, if $a_{12}$ and $a_{21}$ are constant, then we can set $b_{12}(t) = a_{12} + a_{21}(t) + \psi(t)$, $b_{21}(t) = a_{21}(t) + \beta + \psi(t)$, where $\psi$ is a continuous function, $0 \leq \alpha \leq \varepsilon$, $0 \leq \beta \leq \varepsilon$, and $b_{11}(t) = a_{11}(t)$, $b_{22}(t) = a_{22}(t)$. If we choose $\alpha$ and $\beta$ such that $a_{12} - a_{21} + a_{11} - a_{22} = 0$ (the existence of such $\alpha$ and $\beta$ is obvious), then one can show that $B \notin LD_v \cup LD_v \cup L LD_v$. If $a_{12} \notin const$ or $a_{21} \notin const$, then we set $b_{12}(t) = a_{12}(t)$, $b_{21}(t) = a_{21}(t)$, where $0 \leq \alpha \leq \varepsilon$, and $b_{11}(t) = a_{11}(t)$ for all $i, j = 1, 2$, $(i, j) \neq (1, 2)$, or $b_{21}(t) = a_{21}(t)$, where $0 \leq \beta \leq \varepsilon$, and $b_{22}(t) = a_{22}(t)$ for all $i, j = 1, 2$, $(i, j) \neq (2, 1)$, respectively. For both these cases one can show that $B \notin (LD_v \cup LD_v \cup LD_v)$. Indeed, since the sequence $(s_{1i})$ is bounded, there exists a subsequence $(s_{1i_k})$ such that $s_{1i_k} \to s \geq 0$ as $s_{1i_k} \to +\infty$. Without loss of generality, $s_j \to s$ as $i \to +\infty$. So the corresponding values of $t$ we have $[A_{1i}(t), \int_s^t A_{1i}(u)du] = [A_{1i}(t), \int_s^t A_{1i}(u)du] + [A_{2j}(t), \int_s^t A_{2j}(u)du] = [A_{1i}(t), \int_s^t A_{1i}(u)du]$. Since $A_{1i}$ is uniformly bounded on $[0, +\infty]$, we have $[A_{1i}(t), \int_s^t A_{1i}(u)du] \to 0$ as $i \to +\infty$. On the other hand, the sequence $A_{1i}$ is uniformly convergent on the non-negative half-line Therefore $[A_{1i}(t), \int_s^t A_{1i}(u)du] \to [A(t), \int_s^t A_{1i}(u)du]$ as $i \to +\infty$. So the corresponding values of $t$ we have $[A(t), \int_s^t A_{1i}(u)du] \equiv 0$, i.e., $A$ is a bilateral or right Lappo-Danilevskii matrix.

Similarly one can prove

**Theorem 5.** Let $A_{1i} \in LD_v(s_i), i \in \mathbb{N}$, and $\rho(A, A_{1i}) \to 0$ as $i \to +\infty$. If there exist $m, M$ such that $0 < m \leq s_i \leq M < +\infty$ for all $i \in \mathbb{N}$, then $A$ is a left Lappo-Danilevskii matrix.
Theorem 6. There exists a sequence \( A_i, A_k \in LD, (s_i) \in \mathbb{N}, \rho(A, A_{s_i}) \to 0 \) and \( s_i \to +\infty \) as \( i \to +\infty \), such that \( A \notin LD \).

To prove this statement, it is sufficient to consider a sequence \( A_k(t) = a_{ij}(t), i, j = 1, 2 \), such that \( a_{11}(t) = a_{22}(t) = g(t) \) with \( g \) continuous and bounded, and \( a_{21}(t) = \exp(-t), a_{10}(t) \) where:

\[
 f_k = \begin{cases} 
 (1 - \exp(-t)) \exp(-t), & 0 \leq t \leq k, \\
 (1 - \exp(-k)) \exp(-t), & t > k. 
\end{cases}
\]

Theorem 7. There exists a sequence \( A_i, A_j \in LD, (s_i) \in \mathbb{N}, \rho(A, A_{s_i}) \to 0 \) and \( s_i \to +\infty \) as \( i \to +\infty \), such that \( A \notin LD \).

To prove this statement, it is sufficient to consider a sequence \( A_k(t) = a_{ij}(t), i, j = 1, 2 \), such that \( a_{11}(t) = a_{22}(t) = g(t) \) with \( g \) continuous and bounded, and \( a_{21}(t) = \exp(-t), a_{10}(t) \) where:

\[
 f_k = \begin{cases} 
 \exp(-\lambda t - 1 - t), & 0 \leq t \leq k - 1, \\
 \exp(-\lambda k - 1 - t), & t > k - 1. 
\end{cases}
\]

Theorem 8. Let \( A_i \in LD, (s_i) \in \mathbb{N}, \rho(A, A_{s_i}) \to 0 \) as \( i \to +\infty \), then \( A \) is a bilateral Lappo-Danilevskii matrix.

Theorem 9. Let \( A_i \in LD, (s_i) \in \mathbb{N}, \rho(A, A_{s_i}) \to 0 \) as \( i \to +\infty \), then \( A \) is a left Lappo-Danilevskii matrix.

The proofs of Theorem 8 and Theorem 9 are based on the following lemmas.

Lemma 1. Let continuous scalar functions \( f \) and \( g \) satisfy \( f(t) \int_s^t g(u)du = g(t) \times \int_s^t f(u)du \) for some \( s \geq 0 \) and for all \( t \in [b, c] \subset [0, +\infty[. \) If \( \int_s^t g(u)du \neq 0 \) for all \( t \in [b, c] \), then there exists a number \( \lambda \) such that \( \int_s^t f(u)du = \lambda \int_s^t g(u)du \) and \( f(t) = \lambda g(t) \) \( \forall t \in [b, c] \).

Let \( Z(g, s) = \{ t \geq 0 \mid \int_s^t g(u)du = 0 \}, N(g, s) = \{ t \in Z(g, s) \mid g(t) \neq 0 \} \). Denote by \( R(g, s) \) the subset of \( Z(g, s) \setminus N(g, s) \) with the following property: \( \forall t_0 \in R(g, s) \forall \delta > 0 \exists t_1, t_2 < t_0 + \delta, t_0 \notin Z(g, s) \). Denote by \( L(g, s) \) the subset of \( Z(g, s) \setminus N(g, s) \) with the following property: \( \forall t_0 \in L(g, s) \forall \delta > 0 \delta < t_0 < t_0 + \delta, t_0 \notin Z(g, s) \).

Lemma 2. Let continuous scalar functions \( f \) and \( g \) satisfy \( f(t) \int_s^t g(u)du = g(t) \times \int_s^t f(u)du \), for some \( s \geq 0 \) and for all \( t \geq 0 \). Then \( N(g, s) \cup R(g, s) \cup L(g, s) \subset Z(f, s) \).

Lemma 3. Let a sequence of continuous scalar functions \( g_i \) uniformly over \( [0, +\infty[ \) converge to a function \( g \). Then for any \( \sigma \in [0, +\infty[, g(\sigma) \neq 0 \), there exist positive \( c \) and \( \nu \) such that for all \( i > \nu \) the inequalities \( g_i(t) \neq 0, g(t) \neq 0 \) hold for all \( t \in [\sigma - c, \sigma + c] \).

Lemma 4. Let sequences of continuous scalar functions \( g_i \) \( f_i \) uniformly over \( [0, +\infty[ \) converge to functions \( g \) and \( f \) respectively. If for any \( i \in \mathbb{N} \) there is \( s_i \), such that for all \( t \geq 0 \) we have \( f_i(t) \int_s^{s_i} g_i(u)du = g_i(t) \int_s^{s_i} f_i(u)du \), then there exists \( s \geq 0 \) such that the equality \( f(t) \int_s^t g(u)du = g(t) \int_s^t f(u)du \) holds for all \( t \geq 0 \).

References


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