ON THE ENUMERABLE SET OF DIFFERENT CHARACTERISTIC SETS OF SOLUTIONS OF A PFAFFIAN LINEAR SYSTEM

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Consider the Pfaffian linear system

$$\frac{\partial x}{\partial t} = A_i(t)x, \quad x \in \mathbb{R}^n, \quad t = (t_1, t_2) \in R_2^2,$$

with bounded continuously differentiable matrices $A_i(t)$ and $A_2(t)$ satisfying the following condition of complete integrability:

$$\frac{\partial A_1(t)}{\partial t} + A_2(t)A_1(t) = \frac{\partial A_2(t)}{\partial t} + A_1(t)A_2(t), \quad t \in R_2^2.$$

It is well known [1, p. 34] that the ordinary linear system $\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in R_2^2$, with bounded piecewise continuous coefficients has no more than $n$ different characteristic exponents. Let $\lambda|x| = \lambda \in R^2$ be a characteristic vector $[2 - 4]$ of a nontrivial solution $x: R_2^2 \rightarrow R^2 \setminus \{0\}$ of (1) defined by

$$L_\epsilon(\lambda) = \lim_{t \to +\infty} [\ln \|x(t)\| - (\lambda, t)]/\|t\| = 0, \quad L_\epsilon(\lambda - \epsilon e_i) > 0, \quad \forall \epsilon > 0, \quad i = 1, 2.$$

For the characteristic set $\Lambda_x = \bigcup A[x]$ of this solution which is the most natural analog of Lyapunov's characteristic exponent of a one variable vector-function, the essential initial problem about possible number of different characteristic sets $\Lambda_x$ of all nontrivial solutions $x$ of (1) remained open. Note also that the set $\{P_x\}$ of different lower characteristic sets $P_x = \bigcup p[x]$ of all nontrivial solutions $x$ of (1) composed of lower characteristic vectors $[5, 6]$ $p[x] = p \in R^2$ defined by

$$l_\epsilon(p) = \lim_{t \to +\infty} [\ln \|x(t)\| - (p, t)]/\|t\| = 0, \quad l_\epsilon(p + \epsilon e_i) < 0, \quad \forall \epsilon > 0, \quad i = 1, 2,$$

is nonenumerable and, moreover, the set of the lower characteristic vectors $\bigcup_{x \neq 0} P_x$ of (1) has a positive planar Lebesgue measure $[5, 6]$.

It holds the following

**Theorem.** For any sequence $C = \{c_m\}$ of pairwise noncollinear vectors there is a complete integrable two-dimensional system (1) with bounded infinitely differentiable coefficients such that all of its solutions $x(t, c_m), m \in N$, have pairwise different characteristic sets $\Lambda(m)$ with a positive linear Lebesgue measure. If $x(t)$ is a solution of (1) linearly independent with any of $x(t, c_m), c_m \in C$, then its characteristic set $\Lambda_x = \lim_{m \to +\infty} \Lambda(m)$ also has a positive measure.

1. **Construction of the required system.** The preliminary notes. To an enumerable set $C \subseteq R_2^2 \setminus \{0\}$ of the vectors $c_m = (c_{1m}, c_{2m}) \in R^2$ assign the enumerable set $\alpha = \{\alpha_m\} \subseteq R$ of different numbers $\alpha_m \equiv -c_{2m}/c_{1m} \in (-\infty, \infty)$, the ratios of the

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The characteristic sets $\Lambda_x$ and $\Lambda_y$ of the solutions $x \neq 0$ and $y \neq 0$ of (1) are different if $\Lambda_x \bigcap \Lambda_y \neq \Lambda_x \bigcup \Lambda_y$. 
components of the vector \( c_m \). Without loss of generality it can be assumed that first components \( c^{1}_{m} \) of \( c_m \) are nonzero.

In the closed first quarter \( R_{+}^{2} \) of the plane \( R^{2} \) we will build the required Pfaffian system by constructing its fundamental (lower-triangular and infinitely differentiable) system of solutions \( X(t) = (x_{ij}(t))^T \) with \( x_{12}(t) \equiv 0 \) for \( t \in R_{+}^{2} \).

On the interval \(( -\infty, \infty)\) define two infinitely differentiable functions \([7, p. 54]\):

\[
e_{0}(\eta; \eta_{1}, \eta_{2}) = \begin{cases} 
0, & \text{if } \eta_{1} \in (\infty, \eta_{2}), \\
\exp\left(-\left(\eta - \eta_{1}\right)^{2} \right) \exp\left(-\left(\eta - \eta_{2}\right)^{2} \right), & \text{if } \eta_{1} \in (\eta, \eta_{2}), \\
1, & \text{if } \eta_{2} \in (\eta, \infty),
\end{cases}
\]

\[
e_{1}(\eta; \eta_{1}, \eta_{2}) = \begin{cases} 
1, & \text{if } \eta \in (\infty, \eta_{2}), \\
\exp\left(-\left(\eta - \eta_{1}\right)^{2} \right) \exp\left(-\left(\eta - \eta_{2}\right)^{2} \right), & \text{if } \eta \in (\eta, \eta_{2}), \\
0, & \text{if } \eta \in (\eta_{2}, \infty),
\end{cases}
\]

where \(-\infty < \eta_{1} < \eta_{2} < +\infty\) are used for constructing of elements of the matrix \( X(t) \).

With the help of the numbers \( p_{0} = 0, q_{0} = \epsilon \in (0, 1/8), \) and \( q_{k} = 1 - 2^{-k}, p_{k} = q_{k} - 2^{-1-k}, \) \( k \in N, \) define the sectors: the closed ones \( S_k = \{ t \in R_{+}^{2} : q_{k-1} < t_1 < t_2 < q_k \} \) with \( k \geq 0, \) the open ones \( s_k = \{ t \in R_{+}^{2} : q_{k-1} < t_1 < t_2 < q_k \} \) with natural \( k \geq 1, \) and the also sector \( s_0 = \{ t \in R_{+}^{2} : 0 \leq t_1/t_2 < \epsilon \} \).

2. The construction of the diagonal elements of the fundamental system. In \( R_{+}^{2} \) define the positive function \( x_2(t) \) by

\[
\ln x_2(t) = \begin{cases} 
\sqrt{t_2 + t_2/\sqrt{c}} - (\sqrt{t_2} - \sqrt{\epsilon} - \sqrt{t_2/\sqrt{c}})^2 \ln(t_2/t_1; 0, \epsilon), & t \in S_0, \\
\sqrt{t_2 + t_1/\sqrt{c}} - (\sqrt{t_2} - \sqrt{\epsilon} - \sqrt{t_2/\sqrt{c}})^2 \ln(t_2/t_1; 0, \epsilon), & t \in s_0, \\
\sqrt{t_2 + t_1/\sqrt{c}} - (\sqrt{t_2} - \sqrt{\epsilon} - \sqrt{t_2/\sqrt{c}})^2 \ln(t_2/t_1; 0, \epsilon), & t \in R_{+}^{2} \setminus (s_0 \cup S_0) \cap S_k = \emptyset.
\end{cases}
\]

Put the function \( x_1 : R_{+}^{2} \to [1, +\infty) \) be equal to \( x_2 : 1 \) on a closed sector \( S \subset R_{+}^{2}, \) which is bounded by the bisection \( t_2 = t_1 \) and the positive coordinate semiaxis \( t_1 = 0; \) 2) on all sectors \( S_k, k \geq 0. \) In order to define this function on the remaining sectors \( s_k, k \in N, \) we consider the numbers \( r_k = c_{s_{k-1}}, r_0 = 0, k \in N, \) satisfying

\[
(1 + |\alpha_k| + |\alpha_{k+1}|) \exp (\epsilon_{0} - p_{k})^{-2}, \quad k \in N; \quad r_{1} > (1 + |\alpha_{1}|) \exp (\epsilon_{0} - p_{k})^{-2}.
\]

In the sector \( s_k \) we will define \( x_1(t) \) by

\[
\ln x_1(t) = 2\sqrt{t_2} \ln(1 + \epsilon_{0} \ln(\|t\|/r_{1}; 1, 3/2)c_{s_{k-1}}(t_2/t_1; q_{k-1}, p_{k}) - 1), \quad t \in s_k, k \in N.
\]

Note that by definition of the function \( e_{0}(\eta; \eta_{1}, \eta_{2}) \) on the whole axis \( (-\infty, +\infty) \) we have

\[
\ln x_1(t) = 2\sqrt{t_2} \ln(1 + \epsilon_{0} \ln(\|t\|/r_{k}; 1, 3/2)c_{s_{k-1}}(t_2/t_1; q_{k-1}, p_{k}) - 1), \quad t \in s_k, \quad \|t\| \geq 3r_{k}/2.
\]

3. The construction of the off-diagonal elements of the fundamental system. Due to [5, 6], define the off-diagonal element \( x_{21}(t) \) of a constructed two-dimensional linear Pfaffian system with bounded infinitely differentiable coefficients and two-dimensional time by the equality \( x_{21}(t) = x_{2}(t) F(t), t \in R_{+}^{2}, \) where the infinitely differentiable function \( F(t) \) is defined by

\[
F(t) = \begin{cases} 
0, & \text{if } t \in S, \\
\alpha_{k} \ln(\|t\|/r_{k}; 1/2, 1), & \text{if } t \in s_{k}, k \in N, \\
\alpha_{k} \ln(\|t\|/r_{k}; 1/2, 1) + \epsilon_{0} (t_2/t_1; p_{k}) \ln(\|t\|/r_{k}; 1/2, 1) - \epsilon_{0} (t_2/t_1; p_{k}) \ln(\|t\|/r_{k}; 1/2, 1), & \text{if } t \in S, \quad k \geq 0.
\end{cases}
\]

The infinite differentiability of the functions \( x_{21}(t) \geq 1, \) \( x_{2}(t) \geq 1, \) and \( F(t) \) on \( R_{+}^{2} \) follows from the same property of the functions \( e_{0}(t_2/t_1; p_{k}) \) for \( k \geq 0, \epsilon_{0}(\|t\|/r_{k}; 1/2, 1) \) for \( k \geq 1, \) and \( e_{11}(t_2/t_1; q_{k-1}, p_{k}) \) for \( k \in N. \)
4. The boundedness of coefficient matrices

\[ A_i(t) = \frac{\partial X(t)}{\partial t} X^{-1}(t) = \begin{pmatrix}
    x_{1}^t(t) & \frac{\partial x_{2}(t)}{\partial t} \\
    \frac{\partial x_{1}(t)}{\partial t} & x_{2}^{-1}(t) \frac{\partial x_{2}(t)}{\partial t}
\end{pmatrix}, \quad i = 1, 2
\]

of the constructed two-dimensional system (1) is proved by the following statement:

**Lemma.** For all \( m \in \mathbb{N} \) and any \( (\eta_1, \eta_2) \) with the lengths \( \leq 1/2 \) there are the estimates

\[
(\eta - \eta_1)^{-m} e^{a_0(\eta \eta_1, \eta_2)} \leq \sqrt{m/2e} \exp(\eta_2 - \eta_1)^{-2})^{-m}, \quad \eta \in (\eta_1, \eta_2),
\]

\[
(\eta_2 - \eta)^{-m} \exp(-\eta_2 - \eta)^{-2} \leq \sqrt{m/2e}^{-m}, \quad \eta \in (\eta_1, \eta_2).
\]

It is evident, that the infinite differentiability of the matrices \( A_i(t) \) in \( \mathbb{R}^2_+ \) follows from the same property of the nonsingular lower-triangular matrix \( X(t) \). Similarly, the infinite differentiability of the fundamental solutions system \( X(t) \) ensures the feasibility of the complete integrability conditions (2) for the constructed two-dimensional system (1).

5. The construction of the characteristic set of solutions. First for the characteristic set \( \Lambda_2 \) of the solution \( x(t, t_2) = (0, x_2(t)) \) of system (1) we obtain the representation \( \Lambda_2 = \Lambda = [\lambda_1, 1/\lambda_1] \in \mathbb{R}^2_+ \) \( \lambda_1 \in [\sqrt{c_1}, 1/\sqrt{c_1}] \). Then for the solution \( x(t, c_m) \) we establish the relations

\[
\|x(t, c_m)\| = x_1(t) = \|x_2(t)\|^{(1/2) \log \left( s_m - 1/p_m \right)} \equiv \rho_m(t), \quad t \in s_m, \quad \|t\| \geq 3r_m/2;
\]

\[
\max \{x_1(t), \delta k - \alpha_m \|x_2(t)\| \leq \|x(t, c_m)\| \leq (1 + |\alpha_k - \alpha_m|) \|x_2(t)\|, \quad t \in s_k, \quad \|t\| \geq 3r_k/2, \quad k \neq m;
\]

\[
1 \leq \|x(t, c_m)\|/\|x_2(t)\| \leq 1 + |\alpha_k - \alpha_m| + \alpha_{k+1} - \alpha_k, \quad t \in S_k, \quad \|t\| \geq r_{k+1}, \quad k \geq 0;
\]

\[
\|x(t, c_m)\| = \sqrt{1 + \alpha_k^2}, \quad t \in S.
\]

Hence in view of the equality \( \lim_{k \to \infty} \alpha_k = 0 \) the choice of the number \( r_k \), and the uniform in \( t \in s_k \) tending of \( e_1,(l_2/t_1; q_{m-1}, p_k) \) as \( k \to \infty \), it follows that the characteristic set \( \Lambda(m) \) of \( x(t, c_m) \) coincides with the characteristic set of the function \( \rho_m(t) \), which is equal to \( x_2(t) \) outside the sector \( S_m \), \( m \in \mathbb{N} \). By montrival reasonings it established then, that the vector \( \lambda_2(y) \in \mathbb{R}^2 \) with the components

\[
\lambda_2(y) = \varphi_m^1(y), \lambda_1(y) = \varphi_m^2(y) - \psi_m(y) \quad \text{for any} \quad \eta \in [c_1, 1/\sqrt{c_1}],
\]

is located below this hyperbola at \( \lambda_1 \in [\sqrt{c_1}, 1/\sqrt{c_1}] \). In particular, for \( \eta - \eta_m \equiv (q_m - 1 + p_m)/2 \) we obtain the point \( \lambda_2(y) \in \Lambda(m) \) with the coordinates \( \lambda_1(y) = \sqrt{q_m(1 - e^{\gamma_m})}/\sqrt{p_m}, \quad \gamma_m = 10(p_m - q_{m-1})^{-2} \). And the product \( \lambda_1(y) \lambda_2(y) \quad \text{for} \quad \eta \neq \eta_m < 1 \). Obviously, \( \lambda(t) \neq \Lambda(m) \) for any \( t, m \in \mathbb{N} \) \( t \neq m \) and \( \lim_{m \to \infty} \Lambda(m) = \Lambda \). It is not difficult to prove also the equality \( \Lambda_2 = \Lambda \) for a solution (1) linearly independent with any of \( x(t, c_m) \), \( m \in \mathbb{N} \) of the system (1).

The construction of the characteristic sets of all solutions of (1) is completed.

**Remark.** Obviously, from the constructed two-dimensional system (1) it may be possible to obtain an \( n \)-dimensional completely integrable system (1) with bounded infinitely differentiable coefficients in \( \mathbb{R}^2_+ \), which have enumerable number of different characteristic sets of the solutions.

**Problem.** It ought be to clarified, whether the set \( \{A_2\} \) of different characteristic sets \( \Lambda_2 \) of solutions \( x : \mathbb{R}^2_+ \to \mathbb{R}^n \) of a Pfaffian system (1) is finite or enumerable.
REFERENCES


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