ON THE SOLVABILITY OF THE WEIGHTED INITIAL VALUE PROBLEM FOR HIGH ORDER EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper on the basis of the results obtained in [1, 2] optimal, in a certain sense, sufficient conditions for solvability of the weighted initial value problem

\[ u^{(n)}(t) = f(u)(t), \quad u^{(k)}(a) = 0 \quad (k = 0, \ldots, n-1) \]

are established, where \( f \in C^{n-1}([a, b]; \mathbb{R}^m) \rightarrow \mathcal{L}(\mathcal{C}(a, b]; \mathbb{R}^m) \) is a continuous Volterra operator and \( h : [a, b] \rightarrow [0, +\infty) \) is an \((n-1)\)-times continuously differentiable function such that

\[ h^{(k)}(a) = 0 \quad (k = 0, \ldots, n-2), \quad h^{(n-1)}(t) > 0 \quad \text{for} \quad a < t < b. \]

The problem (1), (2) for the case \( n = 1 \) has been investigated in [1, 2]. Therefore below we will assume that \( n \geq 2 \).

Throughout the paper the use will be made of the following notation.

\( \mathbb{R}^m \) is the space of \( m \)-dimensional column vectors \( x = (x_i)_{i=1}^m \) with real components \( x_i \) \((i = 1, \ldots, m)\) and the norm \( \|x\| = \sum_{i=1}^m |x_i| \).

\( \mathbb{R}^n = \{x \in \mathbb{R}^n : \|x\| \leq \rho\} \).

If \( x = (x_i)_{i=1}^m \in \mathbb{R}^m \), then \( \text{sgn}(x) = (\text{sgn} x_i)_{i=1}^m \).

\( x \cdot y \) is the scalar product of the vectors \( x \) and \( y \in \mathbb{R}^m \).

\( C^{n-1}([a, b]; \mathbb{R}^m) \) is the space of \((n-1)\)-times continuously differentiable vector functions \( x : [a, b] \rightarrow \mathbb{R}^m \) with the norm

\[ \|x\|_{C^{n-1}} = \max \left\{ \sum_{k=1}^{n-1} \|x^{(k-1)}(t)\| : \quad a \leq t \leq b \right\}. \]

\( C^{n-1}_h([a, b]; \mathbb{R}^m) \) is the set of \( u \in C^{n-1}([a, b]; \mathbb{R}^m) \) such that

\[ \sup \left\{ \|u^{(k)}(t)\| : \quad a < t \leq b \right\} < +\infty \quad (k = 0, \ldots, n-1). \]

\( C^{n-1}_{h,\rho}([a, b]; \mathbb{R}^m) \) is the set of \( u \in C^{n-1}([a, b]; \mathbb{R}^m) \) satisfying the inequalities

\[ |u^{(k)}(t)| \leq \rho h^{(k)}(t) \quad \text{for} \quad a < t \leq b \quad (k = 0, \ldots, n-1). \]

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If \( x : [a, b] \to \mathbb{R}^m \) is a bounded function and \( a \leq s < t \leq b \) then
\[
\nu(x)(s, t) = \sup \left\{ ||x(\xi)|| : \, s < \xi < t \right\}.
\]

\( L_{\text{loc}}([a, b]; \mathbb{R}^m) \) is the space of vector functions \( x : [a, b] \to \mathbb{R}^m \) which are summable on each segment from \( [a, b] \) with the topology of convergence in the mean on each segment from \( [a, b] \).

**Definition 1.** \( f : C^{n-1}([a, b]; \mathbb{R}^m) \to L_{\text{loc}}([a, b]; \mathbb{R}^m) \) is called a Volterra operator if the equality \( f(x)(t) = f(y)(t) \) holds almost everywhere on \([a, t_0]\) for any \( t_0 \in [a, b] \) and any vector functions \( x \) and \( y \in C^{n-1}([a, b]; \mathbb{R}^m) \) satisfying the condition \( x(t) = y(t) \) for \( a \leq t \leq b \).

**Definition 2.** We will say that the operator \( f : C^{n-1}([a, b]; \mathbb{R}^m) \to L_{\text{loc}}([a, b]; \mathbb{R}^m) \) satisfies the local Carathéodory conditions if it is continuous and there exists a non-decreasing with respect to the second argument function \( \gamma : [a, b] \times [0, +\infty] \to [0, +\infty] \) such that \( \gamma(t, \rho) \in L_{\text{loc}}([a, b]; \mathbb{R}) \) for any \( \rho \in [0, +\infty] \), and the inequality
\[
\|f(x)(t)\| \leq \gamma(t, \|x\|_{C^{n-1}})
\]
is fulfilled for any \( x \in C^{n-1}([a, b]; \mathbb{R}^m) \) almost everywhere on \([a, b]\).

**Definition 3.** If \( f : C^{n-1}([a, b]; \mathbb{R}^m) \to L_{\text{loc}}([a, b]; \mathbb{R}^m) \) is a Volterra operator and \( b_0 \in [a, b] \), then:

(i) for any \( u \in C^{n-1}([a, b]; \mathbb{R}^m) \) by \( f(u) \) is understood the vector function given by the equality \( f(u)(t) = f(u)(t) \) for \( a \leq t \leq b_0 \), where
\[
\mathfrak{M}(t) = \begin{cases} u(t) & \text{for } a \leq t \leq b_0; \\
\sum_{k=1}^{n-1} \frac{(t-b_0)^k}{k!} u^{(k-1)}(b_0) & \text{for } b_0 < t \leq b; 
\end{cases}
\]

(ii) a function \( u \in C^{n-1}([a, b]; \mathbb{R}^m) \) is called a solution of the equation \( (1) \) on the segment \([a, b_0]\) if \( u^{(n-1)} \) is absolutely continuous on each segment contained in \([a, b_0]\) and \( u(t) = f(u)(t) \) almost everywhere on \([a, b_0]\);

(iii) a solution of the equation \( (1) \) on the segment \([a, b]\), satisfying the initial conditions \( (2) \) is called a solution of the problem \( (1) \rightarrow (2) \) on the segment \([a, b]\).

**Definition 4.** The problem \( (1) \rightarrow (2) \) is said to be locally solvable (globally solvable) if it has at least one solution on a segment \([a, b]\) \( \neq [a, b] \) on the segment \([a, b]\).

In what follows, we will assume that \( f : C^{n-1}([a, b]; \mathbb{R}^m) \to L_{\text{loc}}([a, b]; \mathbb{R}^m) \) is a continuous Volterra operator satisfying the local Carathéodory conditions.

**Theorem 1.** Let there exist a positive number \( \rho \) and summable functions \( p_k : [a, b] \to [0, +\infty] \) \( (k = 0, \ldots, n-1) \) and \( q : [a, b] \to [0, +\infty] \) such that
\[
\lim_{t \to a} \frac{1}{H(n-1)} \int_a^t p_k(s) \, ds < 1, \quad \lim_{t \to a} \frac{1}{H(n-1)} \int_a^t q(s) \, ds = 0 \quad (4)
\]
for any \( u \in C^{n-1}([a, b]; \mathbb{R}^m) \) the inequality
\[
f(u)(t) \cdot \nu(u^{(n-1)})(t) \leq \sum_{k=0}^{n-1} p_k(t) \frac{u^{(k)}(t)}{H(k)} (a, t) + q(t)
\]
is fulfilled almost everywhere on \([a, b]\). Then the problem \( (1) \rightarrow (2) \) is locally solvable.

**Proof.** For any \( x \in C([a, b]; \mathbb{R}^m) \) assume
\[
w(x)(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} x(s) \, ds, \quad \bar{f}(x)(t) = f(w(x))(t).
\]
Then by (3)
\[ h(t) = w(h^{(n-1)})(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} h^{(n-1)}(s) \, ds. \tag{7} \]

Obviously, \( \tilde{f} : C([a, b] \setminus \mathbb{R}^m) \to L_{\text{loc}}([a, b] ; \mathbb{R}^m) \) is a continuous Volterra operator satisfying the local Carathéodory conditions.

Assume now that \( y \in C([a, b] ; \mathbb{R}^m), \|y\|_{C} \leq \rho \) and
\[ u(t) = w(h^{(n-1)}y(t)). \tag{8} \]

Then by virtue of (6) and (7)
\[
\left\| u^{(k)}(t) \right\| = \left\| \frac{1}{(n-2-k)!} \int_a^t (t-s)^{n-2-k} h^{(n-1)}(s) y(s) \, ds \right\| \leq \frac{1}{(n-2-k)!} \left( \int_a^t (t-s)^{n-2-k} h^{(n-1)}(s) \, ds \right) \nu(y)(a, t) = h^{(k)}(t) \nu(y)(a, t) \text{ for } a < t \leq b \quad (k = 0, \ldots, n-2),
\]
and
\[ u^{(n-1)}(t) = h^{(n-1)}(t)y(t), \quad \left\| u^{(n-1)}(t) \right\| \leq h^{(n-1)}(t)\nu(y)(a, t) \text{ for } a < t \leq b. \]

Therefore
\[ u \in C^{n-1}(a, b), \quad \text{sgn}(u^{(n-1)}(t)) = \text{sgn}(y(t)), \tag{9} \]
\[ \nu\left( \frac{u^{(k)}}{h^{(k)}} \right)(a, t) \leq \nu(y)(a, t) \text{ for } a < t \leq b \quad (k = 0, \ldots, n-1). \tag{10} \]

On the basis of the conditions (5), (6) and (8)-(10), almost everywhere on \([a, b]\), the inequality
\[
\tilde{f}(h^{(n-1)}y)(t) \cdot \text{sgn}(y(t)) = f(u)(t) \cdot \text{sgn}(u^{(n-1)}(t)) \leq \sum_{k=0}^{n-1} p_k(t) \nu\left( \frac{u^{(k)}}{h^{(k)}} \right)(a, t) + q(t) \leq \sum_{k=0}^{n-1} p_k(t) \nu(y)(a, t) + q(t),
\]
is fulfilled, that is,
\[
\tilde{f}(h^{(n-1)}y)(t) \text{sgn}(y(t)) \leq p(t)\nu(y)(a, t) + q(t), \quad \text{where} \quad p(t) = \sum_{k=0}^{n-1} p_k(t).
\]

On the other hand, it follows from (4),
\[
\limsup_{t \to a} \left( \frac{1}{h^{(n-1)}(t)} \int_a^t p(s) \, ds \right) < 1.
\]

Hence all the conditions of Theorem 2.1 from [1] are fulfilled for the problem
\[
\frac{dx(t)}{dt} = f(x)(t), \quad \lim_{t \to a} \frac{\pi(t)}{h^{(n-1)}(t)} = 0. \tag{11}
\]

Therefore this problem is locally solvable.

Let \( x \) be a solution of the problem (11) on a segment \([a, b_0] \), and \( u(t) = w(x)(t) \).

Then, owing to (6), the function \( u \) is a solution of the problem (1), (2) on \([a, b_0] \). □

Applying Corollary 1 of [2] and repeating the arguments used in proving Theorem 1, we convince ourselves that the following theorem is valid.
Theorem 2. Let for any $u \in C^{n-1}_h([a, b]; \mathbb{R}^n)$ the inequality
\[
 f(u(t)) \cdot \text{sgn}(u^{(n-1)}(t)) \leq \sum_{k=0}^{n-1} p_k(t, p_0(u(t)))\varphi \left( \frac{u^{(k)}}{h^{(k)}} \right)(a, t) + q(t, p_0(u(t)))
\]
be fulfilled almost everywhere on $[a, b]$, where
\[
p_0(u(t)) = \sum_{j=0}^{n-1} p_j(u(t))w_j(t) - \alpha(t),
\]
where $\tau : [a, b] \to [a, b]$ is a continuous function, $p_k (k = 0, \ldots, n - 1)$ and $q : [a, b] \to [0, +\infty]$ are summable with respect to the first argument and continuous and nondecreasing with respect to the second argument. Let furthermore $\tau(t) < t$ for $a < t \leq b$ and
\[
\limsup_{t \to a^+} \left( \frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_{a}^{t} p_k(s, \rho) \, ds \right) < 1, \quad \lim_{t \to a^-} \left( \frac{1}{h^{(n-1)}(t)} \int_{a}^{t} q(s, \rho) \, ds \right) = 0
\]
for some positive constant $\rho$. Then the problem (1), (2) is globally solvable.

A particular case of the equation (1) is the vector differential equation with delay
\[
u^{(m)}(t) = f_0(t, u(\tau_{10}(t)), \ldots, u^{(n-1)}(\tau_{1n-1}(t)), \ldots, u(\tau_0(t)), \ldots, u^{(n-1)}(\tau_{1n-1}(t))),
\]
where $f_0 : [a, b] \times \mathbb{R}^{m} \to \mathbb{R}^{m}$ satisfies the local Carathéodory conditions, and $\tau_{jk} : [a, b] \to [a, b]$ are measurable functions such that $\tau_{jk}(t) \leq t$ for $a \leq t \leq b$ ($i = 1, \ldots, l; k = 0, \ldots, n - 1$).

Theorems 1 and 2 result in the following

Corollary 1. Let $\tau_{1n-1}(t) \equiv t$ and there exist a positive number $\rho$, summable functions $p_{ik} : [a, b] \to [0, +\infty]$ ($i = 1, \ldots, l; k = 0, \ldots, n - 1$) and $q : [a, b] \to [0, +\infty]$ such that
\[
\limsup_{t \to a^+} \left( \frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \sum_{i=1}^{l} \int_{a}^{t} p_{ik}(s) \, ds \right) < 1, \quad \lim_{t \to a^-} \left( \frac{1}{h^{(n-1)}(t)} \int_{a}^{t} q(s) \, ds \right) = 0.
\]
Let furthermore the inequality
\[
f_0(t, h(\tau_{i0}(t))x_{i0}, \ldots, h^{(n-1)}(\tau_{in}(t))x_{in-1}), \ldots
\]
\[
h(\tau_{01}(t))x_{01}, \ldots, h^{(n-1)}(\tau_{n1}(t))x_{n1-1}) \cdot \text{sgn}(x_{in-1}) \leq \sum_{k=0}^{n-1} \sum_{i=1}^{l} p_{ik}(t)||x_{ik}|| + q(t)
\]
be fulfilled on $[a, b] \times \mathbb{R}^{m}$. Then problem (12), (2) is locally solvable.

Corollary 2. Let there exist a number $l_0 \in \{1, \ldots, l - 1\}$ and a continuous function $\tau : [a, b] \to [a, b]$ such that $\tau_{0l-1}(t) \equiv t$
\[
\tau_{jk}(t) \leq \tau(t) < t \quad \text{for} \quad a < t \leq b \quad (i = l_0 - 1, \ldots; \quad k = 0, \ldots, n - 1)
\]
and let the inequality
\[
f_0(t, h(\tau_{i0}(t))x_{i0}, \ldots, h^{(n-1)}(\tau_{in}(t))x_{in-1}), \ldots
\]
\[
h(\tau_{01}(t))x_{01}, \ldots, h^{(n-1)}(\tau_{n1}(t))x_{n1-1}) \cdot \text{sgn}(x_{in-1}) \leq
\]
\[
\leq \sum_{k=0}^{n-1} \sum_{i=1}^{l_0} p_{ik}(t) \left( \sum_{j=0}^{n-1} \sum_{i=i_0+1}^{l} ||x_{ij}|| \right) + q(t) \left( \sum_{j=0}^{n-1} \sum_{i=i_0+1}^{l} ||x_{ij}|| \right)
\]
be fulfilled on $[a, b] \times \mathbb{R}^{m,n}$, where the functions $p_k : [a, b] \times [0, +\infty] \to [0, +\infty]$ (i = 1, ..., $l_1$; $k = 0, \ldots, n - 1$), $q : [a, b] \times [0, +\infty] \to [0, +\infty]$ are summable with respect to the first argument and continuous and nondecreasing with respect to the second argument. Let furthermore

$$\limsup_{t \to a} \left( \frac{1}{h(n-1)} \sum_{k=0}^{n-1} \sum_{i=1}^{l_1} \int_0^t p_k(s, \rho) \, ds \right) < 1, \quad \lim_{t \to a} \left( \frac{1}{h(n-1)} \int_0^t q(s, \rho) \, ds \right) = 0$$

for some positive constant $\rho$. Then problem (12), (2) is globally solvable.

**Remark 1.** Under the conditions of the above-mentioned propositions the right sides of differential equations may have singularities of arbitrary orders. Indeed, as an example let us consider on the interval $[a, b]$ the scalar differential equation

$$u^{(n)}(t) = \sum_{k=0}^{n-1} \left[ \frac{\alpha_k}{\Gamma(\lambda - k)} (\mu^k) + \frac{\beta_k}{\Gamma(\lambda - k)} \gamma_k \right] u^{(k)}(t) + ct^\lambda a_n$$

with the initial conditions

$$\lim_{t \to a} \frac{u^{(k)}(t)}{t^{\lambda - k}} = 0 \quad (k = 0, \ldots, n - 1)$$

where $\mu \in [0, 1]$, $\alpha_k$ and $\beta_k \in \mathbb{R}$, $\mu_k > 1$, $\gamma_k > 1$, $c \in \mathbb{R}$, $\lambda_0 > \lambda$, $g_k : [0, \infty] \times \mathbb{R}^n \to [0, +\infty]$ are continuous functions. By Corollary 2, for the global solvability of problem (13), (14) it is sufficient that

$$\sum_{k=0}^{n-1} \frac{\Gamma(\lambda - k) \cdot \ldots \cdot (\lambda - n + 1)}{\Gamma(\lambda - k)} < 1.$$

**Remark 2.** There exists an example which shows that condition (4) in Theorem 1 is optimal and it cannot be replaced by the condition

$$\limsup_{t \to a} \left( \frac{1}{h(n-1)} \sum_{k=0}^{n-1} \int_0^t p_k(s) \, ds \right) \leq 1.$$

**References**


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