Consider the system of differential equations

\[ u_1'(t) = \sum_{i=1}^{m} p_i(t) u_2(\sigma_i(t)), \quad u_2'(t) = -\sum_{i=1}^{m} q_i(t) u_1(\tau_i(t)), \]

(1)

where \( p_i, q_i \in L_{loc}(\mathbb{R}^+; \mathbb{R}^+), \sigma_i, \tau_i \in C(\mathbb{R}^+; \mathbb{R}^+), \sigma_i(t) \leq t \) for \( t \in \mathbb{R}^+ \), \( \lim_{\rightarrow +\infty} \tau_i(t) = +\infty \), \( \lim_{\rightarrow +\infty} \sigma_i(t) = +\infty \), \( i = 1, \ldots, m \).

Let \( t_0 \in \mathbb{R}^+ \) and \( a_0 = \inf_{t \in [t_0, +\infty]} \{\min[\tau_i(t), \sigma_i(t) : i = 1, \ldots, m] \} \). A continuous vector-function \( (u_1, u_2) \) defined on \([a_0, +\infty[ \) is said to be a proper solution of the system (1) in \([0, +\infty[ \) if it is absolutely continuous on each finite segment contained in \([0, +\infty[ \), satisfies (1) almost everywhere on \([0, +\infty[ \), and \( \sup \{h_1(s) + h_2(s) : s \geq t \} > 0 \) for \( t \geq t_0 \).

A proper solution \((u_1, u_2)\) of the system (1) is said to be oscillatory if both \( u_1 \) and \( u_2 \) have sequences of zeros tending to infinity. If there exists \( t_* \in \mathbb{R}^+ \) such that \( u_1(t_*)u_2(t_*) \neq 0 \) for \( t \geq t_* \), then \((u_1, u_2)\) is said to be nonoscillatory.

In this paper, sufficient conditions are given for the oscillation of proper solutions of the system (1) which make the results contained in [1, 2] more complete.

In the sequel, we will use the following notation: \( p(t) = \sum_{i=1}^{m} p_i(t), q(t) = \sum_{i=1}^{m} q_i(t), \]

\( h(t) = \int_{0}^{t} p(s) ds, h_0(t) = \min \{h(t), h(\tau_i(t)) : i = 1, \ldots, m \} \).

**Theorem 1.** Let

\[ \int_{t_0}^{+\infty} p(t) dt = +\infty, \quad \int_{t_0}^{+\infty} h_0(t) q(t) dt = +\infty, \]

(2)

and there exist a nonincreasing function \( \sigma \in C(\mathbb{R}^+; \mathbb{R}^+) \) such that \( \sigma_i(t) \leq \sigma(t) \leq t \) \( (i = 1, \ldots, m) \) and

\[ \lim_{t \rightarrow +\infty} \sup_{\rightarrow +\infty} h(\sigma(t))/h(t) < +\infty. \]

(3)

If, moreover, there exists \( \varepsilon > 0 \) such that for any \( \lambda \in [0, 1] \)

\[ \lim_{t \rightarrow +\infty} \inf_{t \rightarrow +\infty} \left( h^3(\varepsilon(\sigma(t))) \int_{\tau_i(\sigma(t))}^{+\infty} p(s) h^{-\varepsilon}(s) g(s, \lambda) ds \right) \]

(4)

where \( \varepsilon(\tau_i) = \max \{\max \{\tau_i(s), \eta(s) : i = 1, \ldots, m\} : 0 \leq s \leq t\}, \eta(t) = \sup \{s : \sigma(s) < t\}, \]

\( g(t, \lambda) = \int_{t}^{t}(\xi) \sum_{i=1}^{m} q_i(\xi) h^3(\tau_i(\xi)) d\xi, \)

then every proper solution of the system (1) is oscillatory.

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Theorem 2. Let the conditions (2), (3) be fulfilled, where the function \( \sigma \in C(\mathbb{R}_+; \mathbb{R}_+) \) is nondecreasing, \( \sigma(t) \leq t \) for \( t \in \mathbb{R}_+ \), \( i = 1, \ldots, m \). If, moreover, there exists \( \varepsilon > 0 \) such that for any \( \lambda \in [0, 1] \)

\[
\liminf_{t \to +\infty} h^{-\lambda}(t) \int_0^{\sigma(t)} h(\xi) \sum_{i=1}^{m} q_i(\xi) h^\lambda(\tau_i(\xi)) d\xi > 1 - \lambda + \varepsilon,
\]

then every proper solution of the system (1) is oscillatory.

Theorem 3. Let

\[
\limsup_{t \to +\infty} h^{-1}(\tau_i(t))/h(t) < +\infty \quad (i = 1, \ldots, m)
\]

and there exist \( \varepsilon > 0 \) such that for any \( \lambda \in [0, 1] \)

\[
\liminf_{t \to +\infty} h^{-1}(t) \int_0^{\tau_i(t)} h^2(\xi) \sum_{i=1}^{m} q_i(\xi) [h(\tau_i(\xi))/h(t)]^\lambda d\xi > \lambda(1 - \lambda) + \varepsilon.
\]

Then every proper solution of the system (1) is oscillatory.

Corollary 1. Let (4) be fulfilled and \( \alpha_i \in ]0, +\infty[ \) \( (i = 1, \ldots, m) \), where

\[
\alpha_i = \liminf_{t \to +\infty} h^{-1}(\tau_i(t))/h(t) \quad (i = 1, \ldots, m).
\]

If, moreover, there exists \( \varepsilon > 0 \) such that for any \( \lambda \in [0, 1] \)

\[
\liminf_{t \to +\infty} h^{-1}(t) \int_0^{\tau_i(t)} h^2(s) \sum_{i=1}^{m} \alpha_i^\lambda q_i(s) ds > \lambda(1 - \lambda) + \varepsilon,
\]

then every proper solution of the system (1) is oscillatory.

Corollary 2. Let (4) be fulfilled, \( \alpha_i \in ]0, +\infty[ \), \( q_i(t) \geq q_0(t) \) for \( t \in \mathbb{R}_+ \), \( i = 1, \ldots, m \), where \( q_0 \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+) \), \( \alpha_i \) \( (i = 1, \ldots, m) \) are defined by (5). Then the condition

\[
\liminf_{t \to +\infty} h^{-1}(t) \int_0^{\tau_i(t)} h^2(s) q_0(s) ds > \max \left\{ \lambda(1 - \lambda) \left( \sum_{i=1}^{m} \alpha_i^\lambda \right)^{-1} : \lambda \in [0, 1] \right\}
\]

is sufficient for the oscillation of every proper solution of the system (1).

Corollary 3. Let \( q_0 \in L_{loc}(\mathbb{R}_+; \mathbb{R}_+) \), \( \alpha \in [0, 1] \) and \( \liminf_{t \to +\infty} t^{-1} \int_0^{t} t^{\alpha} q_0(s) ds > 0 \). Then every proper solution of the equation \( u^{(i)} + q_0(t) u(t^n) = 0 \) is oscillatory.

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References


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