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COMPLEX ANALYSIS METHODS IN THE THEORY OF INFINITESIMAL BENDINGS OF SURFACES WITH A FLAT POINT

Abstract. Using I. Vekua's analytic methods, the problem of one-to-one correspondence between infinitesimal bendings of surfaces with a flat point is studied.

1. Introduction. In the paper the objects of study are two surfaces $S_0$ and $S$ given in a rectangular Cartesian coordinate system $Ox_1y_1z_1$ by the equations $S_0: z_1 = (x_1^2 + y_1^2)^{n/2}$, $S: z_1 = (x_1^2 + y_1^2)^{n/2} f(x_1, y_1)$. It is assumed that $S_0$ and $S$ are defined in a domain $G_1$, $(0,0) \in G_1$, $f(x_1, y_1) \in C^3(G_1)$, $f(0,0) > 0$, $n(n > 2)$ is any real number, and for all points of $G_1$ other than $(0,0)$, the gaussian curvature of $S$ is positive.

It is clear that the point $(0,0)$ is a flat one on the surfaces $S_0$ and $S$. At it not only the Gaussian curvature but also all coefficients of the second quadratic forms of $S_0$ and $S$ vanish. At this point the surfaces have with their tangent planes a contact order greater than 1. We call the surface $S_0$ model with respect to $S$ as it is a particular case of $S$ and can be obtained from $S$ under the condition $f(x_1, y_1) = 1$.

The aim of the paper is to establish the following result.

Theorem 1. There exists a one-to-one correspondence between the sets of continuous infinitesimal bendings of the surfaces $S_0$ and $S$.

2. An equivalent analytic problem [1]. We extend the I. Vekua analytic methods on investigating infinitesimal bendings of the above surfaces [2]. On $S_0$ and $S$, we introduce a conjugate isometric parametrization $z = x + iy$, $\bar{z} = -1$. Then infinitesimal bendings of these surfaces will be characterized by the functions $\Phi(z) = z^2 K^{1/4}_0(z)(\delta M_0 + i\delta L_0)$, $w(z) = z^2 K^{1/4}(z)(\delta M + i\delta L)$, where $K_0(z)$ and $K(z)$ are the Gaussian曲率

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curvatures of $S_0$ and $S$, and $\delta M_0, \delta L_0$ and $\delta M, \delta L$ are variations of the coefficients of the second quadratic forms of $S_0$ and $S$, respectively.

In the domain $G$ which is the image of the domain $G_1$ by the mapping $z = z(x_1, y_1)$, these functions satisfy the following generalized Cauchy–Riemann systems:

\begin{align}
2\pi i \Phi - b(0)\overline{\Phi} &= 0, \\
2\pi i w - b(z)\overline{w} &= 0.
\end{align}

Here the singular point $z = 0$ belongs to the domain $G$, $b(0) = (n-2)/2\sqrt{n-1}$ and $b(z)$ is a continuous function in $G$ satisfying $|b(z) - b(0)| < M|z|^{\alpha}$ at least in a sufficiently small neighbourhood of $z = 0$; $M, \alpha$ are positive constants.

Continuous solutions of the systems (1) and (2) are connected by the two-dimensional integral equation

$$w(z) = \Phi(z) + P_G w,$$

where $P_G = S_G \left( \frac{b(z) - b(0)}{2\pi} w(z) \right)$ and

$$S_G f = -\frac{1}{\pi} \int_G \left[ \frac{\Omega_1(z, \zeta)}{\zeta} f(\zeta) + \frac{\Omega_2(z, \zeta)}{\zeta} \overline{f(\zeta)} \right] d\zeta d\eta.$$

Here $\zeta = \xi + i\eta$ and $\Omega_1, \Omega_2$ are certain functions presented in [1]. It is necessary to note that $P_G$ is a completely continuous operator mapping the class $C(G)$ of continuous functions in itself. According to Fredholm’s alternatives, the equation (3) will be uniquely solvable, and consequently a one-to-one correspondence between continuous infinitesimal bendings of the surfaces $S_0$ and $S$ will exist if the following assertion takes place.

**Theorem 2.** The homogeneous equation

$$w^+(z) = P_G w^+,$$ \quad $z \in G,$

in the class $C(G)$ has only the zero solution.

**Scheme of proof of Theorem 2.** Suppose that the equation (4) has a non-trivial solution $w^+(z), z \in G$. Let us note some of its properties. First, we can check that any continuous solution $w^+(z)$ of the equation (4) belongs to the class $D_{1,p}(G), p > 2,$ and satisfies the equation (2). In this case, as was shown in [2] (see Theorem 1.1, p.74), we have

$$w^+(z) = O(|z|^{\text{bl}(p)}) \quad \text{as} \quad z \to 0,$$

From (4) it follows that $w^+(z)$ is continuously extended to the domain $E \setminus \overline{G}$ ($E$ is the $z$–plane and $G$ is the closure of $G$) by a continuous function $w^-(z), z \in E \setminus \overline{G},$ satisfying the equation (1). According to the theory of elliptic
systems, $w^-(z)$ is an analytic function in $E \setminus \overline{G}$ with respect to $z$ and $\overline{G}$. In addition, it is established that

$$|w^-(z)| < M\|b(z) - b(0)\|_{L_1}|z|^{-|b(0)|},$$

(6)

where $M = M(|b(0)|, R_0)$ is a constant depending on $|b(0)|$ and $R_0$ (a maximal distance from $z = 0$ to the boundary of $G$), and $\| \cdot \|_{L_1}$ denotes the norm of a function $f(z)$ in the space $L_1(\overline{G})$.

Thus, on the plane $z$ the continuous function $W(z) = \begin{cases} w^+(z), & z \in \overline{G}, \\ w^-(z), & z \in E \setminus \overline{G} \end{cases}$ is defined. This function is subject to the conditions (5), (6) and satisfies the equation

$$2\pi\partial_z W - B(z)\overline{W} = 0,$$

(7)

in which

$$B(z) = \begin{cases} b(z) & \text{for } z \in \overline{G}, \\ b(0) & \text{for } z \in E \setminus \overline{G}. \end{cases}$$

We establish the following.

**Lemma 1.** If $W(z)$ is a function satisfying the above properties, then $W(z) \equiv 0$, $z \in E$.

From this lemma it follows that $w^+(z) = 0$, $z \in G$ and therefore Theorems 1 and 2 are proved.

4. **Generalization.** Now we consider a surface given in Cartesian coordinates $Ox_1y_1z_1$ by the equation $z_1 = \sum_{k=0}^{n} a_{k,n-k}x_1^{k}y_1^{n-k} + R(x_1, y_1)$, where $n$ ($n \geq 3$) is an integer and $a_{k,n-k}$ are constants. Let $R(x_1, y_1)$ be a sufficiently regular function; moreover, let $R(x_1, y_1) = O((x_1^2 + y_1^2)^{(n+1)/2})$ as $x, y \to 0$. Passing over to polar coordinates ($x_1 = r_1 \cos \varphi$, $y_1 = r_1 \sin \varphi$), we write the surface equation in the form

$$S : \quad z_1 = r_1^n f(\varphi) + R(x_1, y_1),$$

(8)

where $f(\varphi) = \sum_{k=0}^{n} a_{k,n-k}(\cos \varphi)^k(\sin \varphi)^{n-k}$. The requirement of positiveness of the curvature in a neighbourhood of the point $(0, 0)$ imposes on $f(\varphi)$ the restriction

$$-(n - 1) \left( \frac{df}{d\varphi} \right)^2 + nf \frac{df}{d\varphi} + n^2 f^2 > 0.$$

(9)

Besides, we assume $f(\varphi) > 0$.

The first summand in the right side of (8) defines the structure of the surface in a neighbourhood of the flat point. The model surface

$$S_0 : \quad z_1 = r_1^n f(\varphi)$$

(10)
corresponds to it.

Further we will consider the surfaces (8) and (10) under wider assumptions on \( n \) and \( f(\varphi) \). We assume that \( n (n > 2) \) is a real number, \( f(\varphi) \) is a \( 2\pi \)-periodic function from the class \( C^3[0, 2\pi] \), satisfying the inequality (9). It is clear that the surfaces under study are objects with sufficiently general and more complicated structure for a neighbourhood of the flat point \((0, 0)\) compared with those which have been discussed previously.

The problem is to establish a one-to-one correspondence between infinitesimal bendings of surfaces (8) and (10).

As was stated in [3], in a conjugate isometric parametrization \( z = x + iy \) and in terms of complex-valued functions \( \Phi(z) \) and \( w(z) \) introduced earlier, infinitesimal bendings of those surfaces are described by the equations

\[
2\pi \partial_z \Phi - b_0(\varphi) \overline{\Phi} = 0, \\
2\pi \partial_z w - [b_0(\varphi) + B(z)] \overline{w} = 0,
\]

(11) (12)

where the point \( z = 0 \) is interior for the domain \( G \), \( b_0(\varphi) \) is a \( 2\pi \)-periodic continuous function and \( B(z) \) is continuous in \( G \), moreover \( B(z) = O(|z|^\alpha) \) as \( z \to 0, \alpha > 0 \).

Apparently, the model equation (11), seeming simpler in comparison with (12), is nevertheless fairly complicated for investigating. This fact is maybe a main reason why a progress in this respect looks such moderate, see [4, 5], and the problem of correspondence for infinitesimal bendings of the surfaces (8) and (10) remains unsolved.

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