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COPPEL–CONTI SETS OF LINEAR SYSTEMS

Abstract. The paper contains a brief review of results, obtained by the authors, on certain topics in the theory of Coppel-Conti sets of linear systems: the solution of Conti’s problem on the inclusion property as the parameter increases; the construction of criteria for the roughness of these sets and their limit sets under uniformly small or integrable perturbations; applications to the investigation of bounded solutions of perturbed nonhomogeneous linear systems.

We consider the Coppel–Conti sets of linear systems

\[ \dot{x} = A(t)x \quad (1A) \]

with piecewise continuous real coefficients \( A(\cdot) : [0, +\infty) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \), generally speaking, unbounded on the semiaxis \( t \geq 0 \). These sets deal with the problem of boundedness of solutions of the nonhomogeneous linear systems

\[ \dot{y} = A(t)y + f(t) \quad (2) \]

raised in 1930 by O. Perron [1].

System \((1A)\) can be identified with its matrix \( A(\cdot) \) and for convenience will be referred to as system \( A \).
We investigate the sets $L^pS$ of linear systems $$(1_A)$$ with Cauchy matrix $X_A(t, s)$ satisfying

$$C_p(A) \equiv \sup_{t \geq 0} \int_{0}^{+\infty} \|X_A(t, s)\|^p ds < +\infty. \tag{3}$$

The sets $L^pS$ were introduced by W. Coppel [2] for $p = 1$ and by R. Conti [3] for $p > 1$. In these cases they are well studied. It was proved in the cited papers that for $p \geq 1$ the inclusion $A \in L^pS$ is equivalent to the boundedness of all solutions of system (2) with any piecewise continuous nonhomogeneous term $f$ that is bounded (if $p = 1$) or $p/(p-1) - st$ power integrable (if $p > 1$) on the semiaxis $t \geq 0$. We extend the definition of $L^pS$ from $p \geq 1$ to all $p > 0$ by using the condition (3).

R. Conti [4, 5] investigated the sets $L^pS$ as a function of the parameter $p$ and proved [6] that the inclusion $L^pS \subset L^qS$ does not hold for arbitrary $q > p \geq 1$ and posed the following question:

Does the inclusion $L^pS \subset L^qS$ hold for constants $p$ and $q$ such that $p > q \geq 1$?

In our paper [7] we obtained the positive answer to this question.

**Theorem 1 ([7]).** The inclusion $L^pS \subset L^qS$ is valid for all $p > q > 0$.

Therefore, there exist limit sets $\lim_{p \to q \geq 0} L^pS$, $q > 0$, and they satisfy the inclusions

$$\lim_{p \to q \geq 0} L^pS \supset \lim_{p \to q \geq 0} L^qS.$$

Moreover, there is no left or right continuity with respect to the parameter $p > 0$.

Indeed,

$$L^qS \setminus \lim_{p \to q \geq 0} L^pS \neq \emptyset, \quad \lim_{p \to q \geq 0} L^pS \setminus L^qS \neq \emptyset, \quad q > 0$$

(see [6]).

The following criterion for a system $(1_A)$ to belong to the limit set has important applications in investigating the interiors of limit sets.

**Theorem 2 ([8]).** $A \in \lim_{p \to q \geq 0} L^pS$, where $0 < q \leq +\infty$, if and only if

$$2 \min_{\tau \in [T, T+1]} \|X_A(t, \tau)\| \leq 1 \quad \forall t \geq T = T_A \geq 1,$$

$$\lim_{t \to +\infty} \int_{t-	au}^{t} \|X_A(t, s)\|^p ds = 0 \quad \forall p \in (0, q).$$
It was also established in [8] that the following two properties of the constants $C_p(A)$ regarded as functions of the parameter $p > 0$ are satisfied for a fixed $A \in \operatorname{Lim} L^p S$, $q > 0 : 1$) the function $C_{(1)}(A) : (0, +\infty) \to R^+$ is continuous and 2) there exists a system $A \in \operatorname{Lim} L^p S$ such that the function $C_{(1)}(A) : (0, +\infty) \to R^+$ has the characteristic exponent $\lambda[C_{(1)}(A)] = +\infty$.

2. The Structure of the Interior of the Set $L^p S$

We define the interior $\operatorname{Int} L^p S$ of the set $L^p S$ as the set consisting of all $A \in L^p S$ such that $A + Q \in L^p S$ for any piecewise continuous $n \times n$ matrix $Q(t)$ satisfying $\|Q(t)\| < \varepsilon_A$ for all $t \geq 0$ and some $\varepsilon_A > 0$.

**Theorem 3 ([7]).** $\operatorname{Int} L^p S = L^p S$ if and only if $p \geq 1$.

Another Conti problem on the interior of the set $\bigcap_{q > 0} L^p S$ to coincide with the set itself is solved (for $q = +\infty$) by the first of the following two theorems about the properties of the interior of the limit sets.

**Theorem 4 ([8]).** $\operatorname{Int} \lim_{p \to q} L^p S = \lim_{p \to q} L^p S$ if and only if $1 < q \leq +\infty$.

**Theorem 5 ([8]).** $\operatorname{Int} \lim_{p \to q} L^p S = \lim_{p \to q} L^p S$ if and only if $1 \leq q < +\infty$.

We also considered [9] the similar problem whether systems $(1, q)$ and $(1, 2)$ with coefficients close in some integral metric simultaneously belong to either of the sets $L^p S$, $\lim_{\gamma \to p} L^p S$ and $\lim_{\gamma \to p} L^p S$. We obtained the following general result for the integral interior $\operatorname{Int}_q L^p S \equiv \{ A \in L^p S : B \in L^p S, \|B - A\|_q \equiv \{ \int_0^{+\infty} \|B(r) - A(r)\|^q dr\}^{1/q} < +\infty \}, q > 0$, of the set $L^p S$ and for the similar interiors $\lim_{\gamma \to p} L^p S$ and $\lim_{\gamma \to p} L^p S$ of the sets $\lim_{\gamma \to p} L^p S$ and $\lim_{\gamma \to p} L^p S$.

**Theorem 6 ([9]).** $\operatorname{Int}_q M = M$ if and only if
1) $p > 1$ and $q \geq p/(p - 1)$ if $M = L^p S$;
2) $p > 1$ and $q > p/(p - 1)$ if $M = \lim_{\gamma \to p} L^p S$;
3) $p > 1$ and $q \geq p/(p - 1)$ if $M = \lim_{\gamma \to p} L^p S$.

Since inclusions $L^q S \subset L^p S$ are valid for all $q > p > 0$, the similar inclusions $\operatorname{Int} L^p S \subset \operatorname{Int} L^p S$ are valid for their interiors. For the integral interiors $\operatorname{Int}_q L^p S$ with different $q > 0$ but the same $p$ the opposite inclusion is valid, at least for $p \geq 1$. This is given by the following theorem.

**Theorem 7 ([9]).** $\operatorname{Int}_q L^p S \subset \operatorname{Int}_q L^p S$ for $p \geq 1$ and $q < l$. 
The interior \( \text{Int}_q L^p S \) of \( L^p S \), which is clearly a part of the interior \( \text{Int}_{q_0} L^p S \equiv \{ A \in L^p S : A + Q \in L^p S \text{ for any } Q(t) \to 0, t \to +\infty, \text{ and } ||Q||_q < +\infty \} \) of this set for all \( p > 0 \) and \( q > 0 \), does not coincide with the latter for some \( p > 0 \) and \( q > 0 \). The following assertion is valid in the case of small perturbations vanishing at infinity.

**Theorem 8 ([9]).** The interior \( \text{Int}_0 L^p S \) of \( L^p S \) with respect to perturbations \( Q(t) \) vanishing at infinity \((\to 0 \text{ as } t \to +\infty)\) i.e., the set \( \text{Int}_0 L^p S \equiv \{ A \in L^p S : A + Q \in L^p S \text{ for any } Q(t) \to 0 \text{ as } t \to +\infty \} \), coincides for all \( p > 0 \) with the usual interior \( \text{Int} L^p S \).

3. **Some Generalizations**

In this section we consider, instead of a constant \( p > 0 \), a function \( p(t) > 0 \) piecewise continuous for \( t \geq 0 \) and equal at the points of discontinuity to one of its limit values \( p(t+0) > 0 \). We consider two generalizations of the set \( L^p S \) and obtain results for them analogous to Theorem 1 and 2.

First we introduce the set

\[
L^{p(t)}_1 = \left\{ A : \int_0^t ||X_A(t, \tau)||^{p(t)} d\tau \leq c_p(A) \equiv \text{const} < +\infty, \quad t \geq 0 \right\}.
\]

We have the following properties of \( L^{p(t)}_1 S \):

1. \( \bigcup_{p(t) > 0} L^p S \subset \bigcup_{p(t) > 0} L^{p(t)}_1 S \) and \( \bigcup_{p(t) > 0} L^{p(t)}_1 S \setminus \bigcup_{p(t) > 0} L^p S \neq \emptyset \);

2. \( \bigcap_{p(t)} L^{p(t)}_1 S \neq \emptyset \) and \( \bigcap_{p(t) \geq q(t)} L^{p(t)}_1 S \neq \emptyset \) for each fixed \( q(t) > 0 \).

The analog of Theorem 1 for the set \( L^{p(t)}_1 \) is

**Theorem 9 ([7]).** If \( p(t) > 0 \) is piecewise continuous for \( t \geq 0 \) and such that for some \( c > 0, d \geq 0 \) and a measurable set \( M \subset [0, +\infty) \) with

\[
\lim_{t \to +\infty} \text{mes}([\tau, t] \cap M)/(t - \tau) > 0,
\]

the inequality

\[
\sum_{i=1}^k \inf_{\tau \in [0, \Theta] \cap M} \frac{p(t)}{p(t - i\Theta + \tau)} \geq c \ln k - d
\]

holds for the positive integers \( k = 1, \ldots, [t/\Theta] \) and sufficiently large constants \( \Theta > 1 \), then \( L^{p(t)}_1 S \subset L^{p(t)}_1 \) for each piecewise continuous \( q(t) \) such that \( 1 \geq q(t)/p(t) \geq \text{const} > 0, t \geq 0 \).

The conclusion of Theorem 9 holds for:

1. a function \( p(t) \geq \text{const} > 0 \) bounded on the half-line \( t \geq 0 \);
2. a function \( p(t) > 0 \) such that there are constants \( a, b \in (0, 1) \) for which \( p(t)/p(\tau) \geq a \) when \( \tau \in [bt, t] \) and \( t \geq 1 \);
3. a function \( p(t) > 0 \) nondecreasing for \( t \geq 0 \);
4) each power function \( p(t) = t^m \) and a piecewise continuous \( q(t) \) such that \( 1 \geq q(t)/p(t) \geq \text{const} > 0, \ t \geq 1. \)

The structure of the interior of \( L_1^{(t)} S \) for \( p(t) \geq 1 \) is established by the following analog of Theorem 2.

**Theorem 10 ([7]).** The equality \( \text{Int } L_1^{(t)} S = L_1^{(t)} \) holds if and only if there is an interval \([t_0, +\infty)\) on which \( p(t) \) is nonincreasing and not smaller than 1.

Finally we investigate linear-system sets

\[
L_0^{(t)} S = \left\{ A : \int_0^\xi \| X_A(\xi, \tau) \|^p(t) \, d\tau \leq c_p(A) < +\infty, \ 0 \leq \xi \leq t < +\infty \right\}
\]

corresponding to functions \( p(t); \) these sets are clearly empty if \( \lim_{t \to +\infty} p(t) = 0. \)

We have the following inclusions

\[
\bigcup_{p(t) > 0} L^p S \subset \bigcup_{p(t) > 0} L_0^{p(t)} S \subset \bigcup_{p(t) > 0} L_1^{p(t)} S
\]

and each of them is strict.

The properties of these sets ensure that they are nearer to the sets \( L^p S \) than to the \( L_1^{(t)} S. \) The following result corresponding to Theorem 1 holds for \( L_0^{p(t)} S. \)

**Theorem 11 ([7]).** The inclusion \( L_0^{p(t)} S \subset L_0^{q(t)} S \) holds for each \( q(t) \) for which \( 1 \geq q(t)/p(t) \geq \text{const} > 0, \ t \geq t_0. \)

The following assertion distinguishes a difference between properties of \( L_0^{p(t)} S \) and \( L^p S. \)

**Theorem 12 ([7]).** The inclusion \( L_0^{p(t)} S \subset L_0^{q(t)} S \) holds for each function \( q(t) \) such that \( p(t) \geq q(t) \geq \lambda_q \min \{1, p(t)\}, \) where \( \lambda_q = \text{const} \in (0, 1) \) and \( t \geq t_0. \) if and only if \( \lim_{t \to +\infty} p(t) < +\infty. \)

We have the following necessary and sufficient condition for the coincidence of the set \( L_0^{p(t)} S \) with its interior \( \text{Int } L_0^{p(t)} S. \)

**Theorem 13 ([7]).** Int \( L_0^{p(t)} S = L_0^{p(t)} S \neq \emptyset \) if and only if \( p(t) > 0 \) is bounded on the half-line \( t \geq 0 \) and is larger than or equal to 1 on some interval \([t_0, +\infty).\)
4. The Coppel–Conti sets $M^pS$ of Unstable Linear Systems

We also considered the Coppel–Conti sets $M^pS$ of unstable linear systems $(1_A)$ whose Cauchy matrix $X_A(t, \tau)$ satisfies the inequality

$$\int_{t}^{+\infty} \|X_A(t, \tau)\|^p d\tau \leq c_p(A) < +\infty, \quad t \geq 0.$$ 

These sets (if $p \geq 1$) connect with the existence of a unique bounded solution of system $(2)$ for any vector-valued function $f \in L_q[0, +\infty)$ with $q = p/(p - 1)$ conjugate to $p$.

The Coppel problem for these sets is also solved positively.

**Theorem 14 ([10]).** The inclusion $M^qS \subset M^pS$ is valid for all $q > p > 0$.

For the interior $\text{Int } M^pS$ of the set $M^pS$, we have the assertion analogous to Theorem 2.

**Theorem 15 ([10]).** $\text{Int } M^pS = M^pS$ if and only if $p \geq 1$.

5. Linear Systems with $L^p$-dichotomy

Finally we consider the general case of linear systems with an $L^p$-dichotomy. This notion is the extension of the concept of exponential dichotomy [11, 12]. It has been investigated by W.A. Coppel [2, 12], R. Conti [3–6], P. Talpalaru [13], V. Staikos [14] and other authors. It is known [2, 3], that the system $(2)$ has at least one solution bounded on $R^+$ for any $f \in L_q[0, +\infty)$, $q \geq 1$, if and only if the system $(1_A)$ is $L^p$-dichotomous with $1/p + 1/q = 1$.

We extend the definition of $L^p$-dichotomy from $p \geq 1$ to all $p > 0$.

Denote by $X_A(t)$ the fundamental matrix of $(1_A)$, $X_A(0) = E$.

**Definition.** We say that the system $(1_A)$ is $L^p$-dichotomous on $R^+$, $0 < p < +\infty$, and write $A \in L^pD$ if there exist complementary projectors $P_1$ and $P_2$ such that

$$\int_{0}^{t} \|X_A(t)P_1 X_A^{-1}(\tau)\|^p d\tau + \int_{t}^{+\infty} \|X_A(t)P_2 X_A^{-1}(\tau)\|^p d\tau \leq C_p(A) < +\infty, \quad t \geq 0.$$ 

The asymptotic behavior of solutions of an $L^p$-dichotomous system is described by the following lemma (see [2, 15] for $p \geq 1$).

**Lemma 1.** If the system $(1_A)$ is $L^p$-dichotomous with some $p > 0$, then

a) $\lim_{t \to +\infty} x(t) = 0$ for any solution $x(t)$ with $x(0) \in B_1 = P_1 \mathbb{R}^n$,

b) any solution $x(t)$ with $x(0) \in \mathbb{R}^n \setminus B_1$ satisfies $\lim_{t \to +\infty} \|x(t)\| = +\infty$. 

The property of exponential dichotomy is known to be self-dual [11] in the following sense: if a linear system \((1_A)\) is exponentially dichotomous with projectors \(P_1\) and \(P_2\), then the adjoint linear system \(\dot{y} = -A^T(t)y\) is also exponentially dichotomous with projectors \(P_2^T\) and \(P_1^T\). The property of \(L^p\)-dichotomy, however, is not self-dual in this sense.

Lemma 2 ([16]). For any \(p > 0\) there exists an \(L^p\)-dichotomous system such that for any \(q > 0\) the adjoint system is not \(L^q\)-dichotomous.

We obtained [16] that the sets \(L^p D\) satisfy the same narrowing property as its two extreme subsets \(L^p S\) and \(M^p S\) corresponding to the cases \(P_1 = E\) and \(P_1 = 0\), respectively.

Theorem 16 ([16]). Any linear system \(L^p\)-dichotomous with \(p > 0\) is also \(L^q\)-dichotomous with any \(q > 0\) and the same projectors.

This theorem follows from the following criterion for a linear system to be \(L^p\)-dichotomous.

Introduce the sets

\[
\begin{align*}
T^1_\alpha(t) &= \{ \tau \in [0, t] : \|X_A(t)P_1X^{-1}_A(\tau)\| \geq \alpha \}, \\
T^2_\alpha(t) &= \{ \tau \in [t, +\infty) : \|X_A(t)P_2X^{-1}_A(\tau)\| \geq \alpha \}
\end{align*}
\]

for any \(\alpha > 0\).

Theorem 17 ([16]). A linear system \((1_A)\) is \(L^p\)-dichotomous with some \(p > 0\) and projectors \(P_1\) and \(P_2\) if and only if the following conditions are satisfied for some \(\alpha, 0 < \alpha < 1\) :

\[
\text{mes} \left( T^1_\alpha(t) \bigcup T^2_\alpha(t) \right) \leq c(\alpha) < \infty, \quad t \geq 0;
\]

\[
\int_{T^1_\alpha(t)} \|X_A(t)P_1X^{-1}_A(\tau)\|^p d\tau + \int_{T^2_\alpha(t)} \|X_A(t)P_2X^{-1}_A(\tau)\|^p d\tau \leq C < \infty, \quad t \geq 0.
\]

As to the structure of the integral interior of \(L^p D\), we have

Theorem 18 ([13, 16]). If \(p\) and \(q\) are conjugate numbers, then \(\text{Int}_q L^p D = L^p D\).

From here we have the important property of roughness with respect to uniformly small perturbations for the set \(L^1 D\).
6. A LINEAR BOUNDARY VALUE PROBLEM ON $\mathbb{R}^+$

We consider the perturbed nonhomogeneous linear system

$$\dot{y} = F(t)y + g(t)$$

for which we study the following boundary value problem on $\mathbb{R}^+$: the existence and asymptotic behavior of bounded solutions.

Using the foregoing properties of the usual and integral interiors of the Coppel–Conti sets, the inclusion property and the Coppel–Conti theorem [2, 3] we obtain some applications to the above-mentioned boundary problem.

**Theorem 19 ([17]).** Let $F(t) = A(t) + B(t) + D(t)$. $g(t) = f(t) + \varphi(t)$. If $A \in L^pS$ (respectively, $A \in M^pS$) for some $p > 1$, then there exists an $\varepsilon_A > 0$ such that all solutions of the system (5) are bounded (respectively, there exists a unique bounded solution) for any piecewise continuous matrix $B(\cdot)$ with $\|B(t)\| < \varepsilon_A$ for any $t \geq t_B \geq 0$, for any matrix $D(t)$ with $\|D(t)\| \in L_q[0, +\infty)$, $q \geq p/(p-1)$, for any vector function $f(\cdot)$ bounded on the positive semiaxis, and for any $\varphi(\cdot) \in L_q[0, +\infty)$, $q \geq p/(p-1)$.

If $p = 1$, then the matrix $D(\cdot)$ and the function $\varphi(\cdot)$ are to be omitted in this assertion.

If $p > 1$, then a finite sum of the vector-valued functions $\varphi_i(\cdot) \in L_q[0, +\infty)$ with arbitrary $q(i) \geq p/(p-1)$ can be taken for $\varphi(\cdot)$.

In the general case where the system (1.4) is $L^p$-dichotomous, $p \geq 1$, the dimension of the subspace of all bounded solutions of (1.4) coincides with the dimension of the corresponding subspace of the system (1.4.4) if $\|B(t)\| \in L_q[0, +\infty)$ with $q = p/(p-1)$.

It follows

**Theorem 20 ([16]).** Let $F(t) = A(t) + B(t)$. If $A \in L^pD$, $p > 1$, then for any matrix $B(\cdot)$, $\|B(t)\| \in L_q[0, +\infty)$ with $q$ conjugate to $p$, and for any vector-function $g(\cdot) \in L_r[0, +\infty)$ with $r \geq p/(p-1)$, the system (4) has a $k$-parameter family of solutions $y(t)$ such that $\lim_{t \to +\infty} y(t) = 0$, where $k = \text{rank} P_1$.

**References**


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