Consider a linear system
\[
\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0,
\]  
(1.4)
with piecewise continuous bounded coefficients and a bimodal [1, p. 49] system of solutions \(X(t)\) ordered in increasing exponents. Let \(\lambda_i(A)\) be the characteristic exponent of the \(i\)-th column of the matrix \(X(t)\) and \(\delta_i\) be the characteristic exponent of the \(i\)-th row of the matrix \(X^{-1}(t)\). By means of the sums \(\sigma_i(A) = \lambda_i(A) + \delta_i(A)\), we introduce (see [2]) the number \(\sigma_0(A) = \frac{\sigma_i(A) + \sigma_j(A)}{2}\), in which the indices \(m \in \{1, \ldots, n\}\) and \(t \in \{1, \ldots, n\}\), \(i \neq m\), are defined by the equalities \(\sigma_m(A) = \max \{\sigma_i(A)\}\), \(\sigma_i(A) = \max \{\sigma_i(A)\}\).

In [2] it is established that the characteristic exponents \(\lambda_1(A + Q) \leq \cdots \leq \lambda_n(A + Q)\), of the perturbed system (1.4+Q) with a piecewise continuous perturbation \(Q(t)\) whose Lyapunov exponent \(\lambda(Q)\) satisfies \(\lambda(Q) < -\sigma_0(A)\), admit the estimates
\[
\lambda_{\mu(i)}(A + Q) \leq \lambda_i(A) + \frac{\sigma_i(A) - \sigma_j(A)}{2}, \quad i = 1, \ldots, n.
\]  
(2)

The question of attainability of these estimates arises. The following theorem gives the positive answer to it in a rather general case.

**Theorem.** For any numbers \(2 \leq n \in \mathbb{N}\), \(m \in \{1, \ldots, n\}\), \(\lambda_1 \leq \cdots \leq \lambda_m\), \(0 < \sigma_1 < \sigma_2\) and any \(\varepsilon \in (0, (\sigma_2 - \sigma_1)/2)\) satisfying the additional condition \(\varepsilon < \lambda_p - \lambda_m + (\sigma_2 - \sigma_1)/2\) if there is (the least) \(p \in \{1, \ldots, m - 1\}\) for which \(\lambda_m < \lambda_p + (\sigma_2 - \sigma_1)/2\), there exist: (i) a system (1.4) with infinitely differentiable bounded coefficients such that \(\lambda_i(A) = \lambda_i\), \(i = 1, \ldots, n\), \(\sigma_m(A) = \sigma_2\) and \(\sigma_i(A) = \sigma_1\) for \(m \neq i \in \{1, \ldots, n\}\); (ii) an analytical perturbation \(Q(t)\) with the Lyapunov exponent \(\lambda(Q) < -\sigma_0 = -\frac{\sigma_1 + \sigma_2}{2}\) such that the perturbed system (1.4+Q) has all different characteristic exponents and: 1) in the absence of the above specified \(p\), \(\lambda_m(A + Q) = \lambda_m\) and
\[
\lambda_i(A + Q) = \lambda_i + \frac{\sigma_2 - \sigma_1}{2} - \frac{1 + n - i}{n} \varepsilon,
\]  
(3)

for \(m \neq i = 1, \ldots, n\); 2) in the presence of such \(p\), \(\lambda_p(A + Q) = \lambda_m\), \(\lambda_i(A + Q) = \lambda_{i+1} + \frac{\sigma_2 - \sigma_1}{2} - \frac{1 + n - i}{n} \varepsilon\), \(i = p + 1, \ldots, m\), and other exponents determined by the formula (3).

**Proof.** We will construct not the system (1.4) itself but its fundamental system of solutions \(X(t) = \text{diag} \{ \exp \{ t f_1(t) \}, \ldots, \exp \{ t f_n(t) \} \}\). Fix \(\theta > 1\) and a rather small \(\varepsilon > 0\)
satisfying the necessary conditions. On the segments $\nabla \delta(\gamma) \equiv \mid \theta^{k+i}, \theta^{k+1} \mid, k \geq 0$, with the determined below number $\gamma \in (0, 1)$, we define the functions $f_k(t)$ by

$$f_k(t) = \begin{cases} \lambda_i, & t \in \nabla 2^k(\gamma), \\ -\delta_i, & t \in \nabla 2^{k+1}(\gamma), \\ \text{m} \neq i = 1, \ldots, n. \end{cases}$$

On the initial segment $[0, 1]$ let's assume $f_0(t) = -\delta_i, \text{m} \neq i = 1, \ldots, n, f_m(t) = \lambda_m$. On other intervals $\Delta \delta(\gamma) \equiv (\theta^k, \theta^{k+1}), k \geq 0$, the function $f_k(t), i = 1, \ldots, n$, are defined by means of a special infinitely differentiable function

$$f(t; \eta_1, \eta_2, \eta_3) = a + (b-a) \exp \left( -\ln^2(t/\eta_1) \right) \times \exp \left[ -\ln^2(t/\eta_2) \right], \quad \eta_1 < t < \eta_2,$$

updating [3] the standard function from [4, p. 54]. The functions $f_k(t)$ on the interval $\Delta \delta(\gamma), k \geq 0$, are defined by the equality

$$f_k(t) = f(t; \eta_1, \eta_2, \eta_3) = a + (b-a) \exp \left( -\ln^2(t/\eta_1) \right) \times \exp \left[ -\ln^2(t/\eta_2) \right], \quad \eta_1 < t < \eta_2,$$

where $t_0 \equiv \theta^m$. It is easy to see that the system (1) so constructed has infinitely differentiable coefficients with all derivatives bounded.

We construct an $n$-th order matrix of perturbation $Q(t)$ as having nonzero elements only in the $m$-th line, except $q_{mm}(t) = 0, \text{m} \geq 0$. These elements look like

$$q_m(t, \varepsilon_i) = \exp(\epsilon_0 - \varepsilon_i), \quad \varepsilon_i \neq \text{m}, \quad t \geq 0,$$

with specially determined below constant $\varepsilon_i, \text{i} \neq \text{m}$.

We choose $\gamma > 0$ involved in the definition of the system (1) so that

$$2(\lambda_n - \lambda_1) + \sigma_2 - \sigma_1(\theta^\gamma - 1) < 2\epsilon/n.$$

Denote the $i$-th solution of the system (1) by $Y_i(t; \varepsilon_i), \text{i} \neq \text{m}$. Its components are

$$Y_{mi}(t; \varepsilon_i) = 0, \quad \text{j} \neq \text{i}, \text{m} \geq 0; \quad Y_{ii}(t; \varepsilon_i) = x_i(t) + \exp(t f_0(t));$$

$$Y_{mi}(t; \varepsilon_i) = x_{m}(t) \left[ q_{mi}(t; \varepsilon_i \exp \left( -\ln^2(t/\eta_1) \right) \times \exp \left[ -\ln^2(t/\eta_2) \right) \right],$$

where the constant $Y_{mi}(t; \varepsilon_i) = 0, \text{m} \geq 0$, or the Lyapunov exponent $\lambda[q_{mi}(x_1, x_m)]$ of the integrand is not less than zero, and $Y_{mi}(t; \varepsilon_i) = -\int_0^\infty q_{mi}(x_1, x_m) \exp(t f_0(t)) \, dt$. It is obvious that the $m$-th solution of the system (1) the vector-function $Y_{mi}(t)$ with the unique different from zero component $Y_{mi}(t) = x_{mi}(t)$.

For any fixed $i \neq \text{m}$, let's establish now the existence of a constant $\varepsilon_i = \delta_i > 0$ that the corresponding solution $Y_i(t; \varepsilon_i)$ has an exponent $\Lambda[Y_i] = \lambda_i + (\sigma_2 - \sigma_1)/2 - (1 + n - i)/n$.

For this purpose at first we shall establish existence of a constant $\varepsilon_i^{(1)} > 0$ such that the inequality $\lambda[Y_i] > \lambda_1 + (\sigma_2 - \sigma_1)/2 - \epsilon_1/n$ be true. Really, in the case $\lambda + \delta_n > -\epsilon_1/n$, due to the condition (5) a constant $\varepsilon_i^{(1)} > 0$ exists such that $\lambda_1 + \delta_n > -\epsilon_1 > 0$ and for sequence $\{t^{i+1}_k\}$ the following estimates are fulfilled

$$\Lambda[Y_i] \geq \lim_{k \to \infty} -\frac{1}{t^{i+1}_k} \ln \left[ \lambda_1 + (\sigma_2 - \sigma_1)/2 - \lambda_n - \lambda_1 + (\sigma_2 - \sigma_1)/2 - (1 - \theta^\gamma - 1) \varepsilon_i^{(1)} \theta^{-\gamma} \right] \geq \varepsilon_1 + (\sigma_2 - \sigma_1)/2 - \epsilon_1/n.$$
In the case $\lambda_i + \delta_m - \sigma_0 \leq 0$, on the basis of the same condition (3), there exists a constant $\varepsilon_i^{(1)} > 0$ such that for a sequence $\{t_{2k}\}$ the inequalities

$$
\lambda|Y| \geq \lim_{k \to \infty} \frac{1}{t_{2k+1}} \log |Y_m(t_{2k}, \varepsilon_i^{(1)})| \geq \lambda_m + \lim_{k \to \infty} \frac{1}{t_{2k}} \int_{t_{2k+1}}^{t_{2k+2}} q_m x_i x_m^{-1} \, dt = 
$$

$$
= \lambda_m + (\lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(1)})/2 \geq \lambda_i + (\sigma_2 - \sigma_1)/2 + [\lambda_i - \lambda_m - 
$$

$$
- (\sigma_2 - \sigma_1)/2](\theta_0 - 1) - \varepsilon_i^{(1)} > \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon/n. \quad (62)
$$

Let’s establish now the existence of a constant $\varepsilon_i^{(2)} > \varepsilon_i^{(1)}$ such that $\lambda|Y_m(t, \varepsilon_i^{(2)})| < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon$. In the first place, choose this constant $\varepsilon_i^{(2)}$ so large, that the $\lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(2)} < 0$. Then, due to the Lyapunov lemma concerning the exponent of the integral, we have

$$
\lambda|Y_m(t, \varepsilon_i^{(2)})| \leq \lambda_m + \lambda_i + \delta_m - \sigma_0 - \varepsilon_i^{(2)} < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon \quad (7)
$$

under the second additional condition $\varepsilon_i^{(2)} > \varepsilon$ on $\varepsilon_i^{(2)}$. Thus, the inequality $\lambda|Y(t, \varepsilon_i)| < \lambda_i + (\sigma_2 - \sigma_1)/2 - \varepsilon$ becomes obvious. Due to established in [5] continuous dependence of the exponent $\lambda|Y(t, \varepsilon)|$ on the parameter $\varepsilon_i > 0$, from (61), (62) and (7) it follows the existence of the required $\varepsilon_i \in (\varepsilon_i^{(1)}, \varepsilon_i^{(2)})$.

For the completion of the proof of the theorem it is necessary to order the solutions $Y_1(t, \varepsilon_i), i \neq m,$ and $Y_m(t)$ according to increase of their exponents. In the first case $\lambda_m \geq \lambda_{m-1} + (\sigma_2 - \sigma_1)/2$, mentioned in the formulation of the theorem, we have obvious inequalities $\lambda|Y_1| < \cdots < \lambda|Y_{m-1}| < \lambda|Y_m| = \lambda_m < \lambda|Y_{m+1}| < \cdots < \lambda|Y_n|$ and so all characteristic exponents of the perturbed system (1.4+Q) are different. In the second case of existence of the least $p \in \{1, \ldots, m-1\}$ for which $\lambda_m < \lambda_p + (\sigma_2 - \sigma_1)/2$, the fundamental system, ordered in decreasing of the exponents, looks like $Y(t) = Y_1(t), \ldots, Y_{p-1}(t), Y_m(t), Y_p(t), \ldots, Y_{m-1}(t), Y_{m+1}(t), \ldots, Y_n(t)$, and the exponents of its solutions are all various, because of the choice, in this case, of the number $\varepsilon > 0$, and for the obtained exponents $\lambda|Y|$ the inequalities

$$
\lambda|Y_{m+1}| = \lambda_{p-1} + (\sigma_2 - \sigma_1)/2 - (2 + n - p)\varepsilon/n < 
$$

$$
< \lambda|Y_m| = \lambda_p + (\sigma_2 - \sigma_1)/2 - (1 + n - p)\varepsilon/n < 
$$

$$
\cdots < \lambda|Y_n| < \lambda|Y_{m+1}| < \cdots < \lambda|Y_n|
$$

are true. Thus, the fundamental matrix $Y(t)$ is normal and the system (1.4+Q), in this case, has all different characteristic exponents specified in the formulation of the theorem. □

**Remark.** For the characteristic exponents of the systems (1.4) and (1.4+Q) constructed in the proof of the theorem, the attainability of the estimates (2) is shown by the inequalities $\lambda_k |A + Q| \geq \lambda_i |A| + (\sigma_2 - \sigma_1)/2 - \varepsilon$, $i = 1, \ldots, n$, which are valid for the permutation

$$
k(i) = \begin{cases} 
q, & \text{if } i = 1, \ldots, p - 1, \ldots, m + 1, \ldots, n, \\
p, & \text{if } i = m, \\
i + 1, & \text{if } i = p, \ldots, m - 1.
\end{cases}
$$


Author’s addresses:

N. A. Irobov
Institute of Mathematics
Belorussian Academy of Sciences
11, Surganova St., Minsk 220072
Belarus

S. N. Bata
Institute of Mathematics
Belorussian Academy of Sciences
11, Surganova St., Minsk 220072
Belarus