Consider a linear system
\[ \dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \]  
(1.4)
with a piecewise continuous bounded matrix of coefficients \( A(\cdot) \); the characteristic exponents \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \); the incorrectness coefficients of Perron \([1]\) \( \sigma_P(A) \) and Grobman \([1]\) \( \sigma_G(A) \), and with a normal ordered system \( X(t) = [X_1(t), \ldots, X_n(t)] \) of its solutions \( X(t) \). Along with the system (1.4) consider perturbed systems \( (1.4+Q) \) with piecewise continuous Perron perturbations \( Q(\cdot) \) determined by the condition \( \lambda[U] \equiv \lim_{\to \pm \infty} \frac{1}{t} \ln \|Q(t)\| < -\sigma_P(A) \).

For Perron perturbations only three following results are known: 1) the upper exponents \( \lambda_2(A) \) and \( \lambda_2(A+Q) \) of the two-dimensional systems (1.4) and (1.4+Q) respectively, coincide \([1]\); 2) generally speaking, the lower exponents \( \lambda_1(A) \) and \( \lambda_1(A+Q) \) of these systems have not this property \([2]\); 3) all characteristic exponents of three and higher-order systems (1.4) are, generally speaking, unstable \((N. A. Izo\'bov, S. N. Batan)\). Therefore, two-dimensional systems play a special role in the study of the behaviour of their characteristic exponents (lower and upper) under Perron perturbations.

For the two-dimensional system (1.4), introduce the angle \( \gamma(t) \equiv \angle[X_1(t), X_2(t)] \) between the solutions \( X_1(t) \) and \( X_2(t) \) forming its normal system of solutions.

**Theorem 1.** For the lower exponent \( \lambda_1(A+Q) \) of the two-dimensional system under any Perron perturbation \( Q(\cdot) \) the following is true: 1) \( \lambda_1(A+Q) = \lambda_1(A) \) if \( \sigma_P(A) = \sigma_G(A) \); 2) \( \lambda_2(A) > \lambda_1(A+Q) > 2\lambda_1(A) - \lambda_2(A) \) and nonstrict \( \lambda_1(A+Q) \geq \lambda(X_1 \sin \gamma) \) otherwise.

**Scheme of proof.** 1. The equality \( \lambda_1(A+Q) = \lambda_1(A) \) if \( \sigma_P(A) = \sigma_G(A) \) is a consequence of the Grobman theorem.
2. Supposing without loss of generality (1.4) to be a lower-triangular system, we transform (1.4+Q) by \( y = X(t)z \) to
\[ \dot{z} = Q(t)z, \quad z \in \mathbb{R}^2, \quad t \geq 0, \]  
(2)
a system of linear asymptotic balance \( \lambda[U] < 0 \). This allows us to prove the inequalities
\[ \lambda[X_1 \sin \gamma] \leq \lambda_1(A+Q) < \lambda_2. \]  
(3)
3. In the case due to (3) \( \lambda[X_1 \sin \gamma] \leq 2\lambda_1(A) - \lambda_2(A) \) we establish that the Perron lower-triangular perturbation \( Q_T(\cdot) \) preserves the characteristic exponents of the initial system (1.4), as well as of the conjugate one \( (1-A_T) \), and their Perron incorrectness coefficient is invariant. This allows to include the lower-triangular part \( Q_T(\cdot) \) of the
Perron perturbation $Q(t)$ into the very matrix $A(t)$ and to suppose that only the element $q_{12}(t)$ with the exponent $\lambda_{[12]} < -\sigma_P[A]$ of the matrix $Q(t)$ is nonzero.

4. Assume that the inequality $\lambda(A + Q) \leq 2\lambda_1(A) - \lambda_2(A)$, opposite to the order under investigation, is satisfied. Then the second component $y_2(t) = x_{12}(t)z_2(t) + x_{22}(t)z_2(t)$ of the solution $y(t) = X(t)Q(t)$ of (1.4 + $Q$), realizing the lower exponent $\lambda_1(A + Q)$ with the appropriate solution $x(t) = z_2(t)z_2(t)$ of (2) or $|1 - \lambda_2(A)|$ has the exponents of its terms $\lambda[x_{21}, z_1] = \lambda_1(A)$, $\lambda[x_{22}, z_2] = \lambda_1(A)$, and, as a result, $\sigma$ has the exponent $\lambda[y_{22}] = \lambda_1(A + Q) = \lambda_1(A)$, which contradicts to the assumption and to the inequality $\lambda_1(A) < \lambda_2(A)$ following from the condition $\sigma_P(A) < \sigma_C[A]$. ■

The exactness of the bounds of the lower exponent $\lambda(A + Q)$ of (1.4 + $Q$) is established by Theorems 2 and 3 below.

**Theorem 2.** For any numbers $\lambda_1 > 0$, $\sigma_0 \geq 2(\lambda_2 - \lambda_1)$, $a \in (2\lambda_1 - \lambda_2, \lambda_1]$ and $\beta \in [\alpha, \alpha + \lambda_2 - \lambda_1)$ there exists a two-dimensional system (1.4) with an infinitely differentiable bounded matrix of coefficients $A(t)$, the characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, 2$, and the Perron incorrectness coefficient $\sigma_P(A) = \sigma_0$. Moreover, for any $\lambda \in [\alpha, \beta]$ there exists an analytical Perron perturbation $Q_\lambda(t)$ such that the system (1.4 + $Q_\lambda$) has a lower exponent $\lambda_1(A + Q_\lambda)$.

**Scheme of a proof.** 1. Fix numbers $\theta > 1$ and $\theta_0 \in (1, \theta^{1/4})$. Using the points $s_\theta \equiv (\theta_0 t)^k$ and $s_k \equiv (\theta_0 t)^k$, $k \geq 0$, define the functions $f_i(t)$ and $f_i(t)$ on the half-axis $t \geq 0$ as follows. On the segments $[s_k, s_{k+1}]$ with zero and even $k$ define them by $f_i(t) = (i - 1)x_i^0 + (i - 2)\delta_i$, $i = 1, 2$, while with odd $k$ by $f_i(t) = (2 - i)x_i^0 + (1 - i)\delta_i$, $i = 1, 2$. Here $x_i^0 = \lambda_i x$, and the numbers $\delta_1$ and $\delta_2$ satisfy $\lambda_2 + \delta_2 = \lambda_2 + \delta_1 = \sigma_0$. On the intervals $(s_k, t_k)$ put

$$f_i(t) = f_i[s_k, t_k], f_i(t_k) = f_i(t_k) + [f_i(t_k) - f_i(t_k)] = f_i(t_k) + \exp[-\ln^2(t/t_k)] \exp[-\ln^2(t/t_k)], \quad (4)$$

using for this purpose an analogue of the well-known infinitely differentiable function [see B. Gelbaum and J. Olmsted “Counterexamples in Analysis”]; on the interval $[0, 1]$ these functions are continued as the constants $f_i(0)$. It is easy to check that $a_i(t) \equiv d[f_i(t)]/dt$ are bounded and infinitely differentiable.

2. We will build the matrix $A(t)$ of (1.4) as lower-triangular with already defined diagonal coefficients $a_i(t)$ and the off-diagonal coefficient $a_2(t, \sigma) = -e^{-\sigma t}$, $t \geq 0$, with such $\sigma > a_0$ that the second component $x_{21}(t, \sigma)$ of its solution $x(t) = (\exp tf_1(t), x_{21}(t, \sigma))$ has the exponent $\lambda[x_{21}] = \lambda_1$. For the proof of the existence of such $\sigma > 0$ we establish the Lipschitz condition $||\lambda[x_{21}; \sigma_2] - \lambda[x_{21}; \sigma_1]|| \leq q(\sigma_2 - \sigma_1)$ with a constant $q = q(\sigma_0) > 1$, whose exponent $\lambda[x_{21}; \sigma]$ satisfies $\sigma_1, \sigma_2 \geq \sigma_0 + \epsilon_0$, as well as the estimates

$$\lambda_1 + \delta - \varepsilon \geq \lambda[x_{21}; \sigma_0 + \varepsilon] \geq \lambda_1 + (\delta - \varepsilon)\delta_0 - (\lambda_2 - \lambda_1)(\delta_0^2 - 1) \geq \lambda_1 + (\delta - \varepsilon)\delta_0 - (\lambda_2 - \lambda_1)(\delta_0^2 - 1) \quad (5)$$

for all $\varepsilon > 0$ satisfying $\lambda_2 - \lambda_1 > \delta - \varepsilon$, where $\delta = a - (2\lambda_1 - \lambda_2)$. Due to the proved continuity of $\sigma > 0$ of the exponent $\lambda[x_{21}; \sigma]$ and inequalities (5), we obtain now, taking $\theta_0 - 1 > 0$ small enough, the existence of the required $\sigma_1 > \sigma_0 : \lambda[x_{21}; \sigma_1] = \lambda_1$. The required equality $\sigma_P(A) = \sigma_0$ is established via the proof of the inequality $\lambda[x_{21}; (x_1 x_2)] \leq \delta_2$.

3. Let $R(t)$ be an analytical Perron perturbation with the unique nonzero element $b_{21}(t, \sigma) = -a_{21}(t, \sigma) = \exp(-\sigma t)$, $\sigma > \sigma_0$, $t \geq 0$. Arguing as in the part 2, we establish the existence of such number $\sigma_2 = \sigma(\lambda) > \sigma_0$ that the second component of the solution $y(t) = (\exp tf_1(t), y_{21}(t, \sigma))$ of (1.4 + $R$) realizing its lower exponent $\lambda_1(A + R)$ has the exponent $\lambda[y_{21}; \sigma_2] = \lambda$. To complete the proof it is sufficient to put $R(t) = Q_1(t)$.
The condition \( \alpha = \lambda [X_1 \sin \gamma] > 2 \lambda_1 (A) - \lambda_2 (A) \) is carried out for the system (1.4) constructed in the proof of Theorem 2, with normal system of solutions \( X = [X_1, X_2] \). There is the question: what will occur to a lower exponent \( \lambda \) in case of Perron perturbation \( Q ( \cdot ) \) and of the fulfillment of an opposite condition \( \alpha = \lambda [X_1 \sin \gamma] \leq 2 \lambda_1 (A) - \lambda_2 (A) \). The partial answer to it is given by the following

**Theorem 3.** For any numbers \( \alpha < 2 \lambda_1 - \lambda_2 < \lambda_1 < \beta < \lambda_2 \) and \( \sigma_0 > \lambda_2 - \alpha \) there exists a two-dimensional system (1.4) with a bounded infinitely differentiable matrix of coefficients \( A ( \cdot ) \), a normal ordered system of solutions \( X = [X_1, X_2] \) satisfying \( \lambda [X_1 \sin \gamma] = \alpha \), the characteristic exponents \( \lambda_i (A) = \lambda_i \), \( i = 1, 2 \), the Perron incorrectness coefficient \( \sigma_P (A) = \sigma_0 \) and such, that for any \( \lambda \in [\lambda_1, \beta] \) there exists a system (1.4+Q) with infinitely differentiable Perron perturbation \( Q ( \cdot ) \) and lower exponent \( \lambda (A + Q) = \lambda \).

**Scheme of a proof.** 1. Fix a parameter \( \theta \) satisfying \( 1 < \sqrt{\theta} < \min \{ \sigma_0 + (i - 1) \alpha \sigma_0 (\sigma_0 + \alpha - \lambda_2) \} \), \( i = 1, 2 \), and introduce the points \( t_k = \theta^k \), \( t_k = t_k \theta^{k + 1} \), \( t^{+}_{0, k - 1}, t^{\pm}_{0, k + 1}, k \in \mathbb{Z} \). Define the functions \( f_1 (t) = \alpha \) for all \( t \geq 0 \) and \( f_2 (t) \) as follows: \( f_2 (t) = \lambda - \sigma_0 = - \alpha \) for \( t \in [0, c_k] \), \( i = 0, 1, 2, 3 \) and \( t^{+}_{0, k + 1}, t^{\pm}_{0, k + 1}, k \geq 0 \), on all segments \( [t^{+}_{1, k + 1}, t^{+}_{1, k + 1}] \) and \( f_2 (t) = c_k \) for \( t \in [t^{+}_{1, k + 1}, t^{+}_{1, k + 1}] \). The partial answer to it is given by the representation of \( \lambda (A + Q) \).

2. Using the form of the constructed functions \( a_2 (t) \) and \( a_2 (t) \), for the second component \( x_2 (t) \) of the solution \( x (t) \) of the solution lower-triangular system (1.4) with initial value \( x (0) = (1, 0) \) we establish the existence of a bounded sequence \( (c_k) \), the numbers \( c_k \) are used in the definition of \( f_2 (t) \) on the intervals \( [t_k, t_k + 1] \), that the inequalities \( x_2 (t_k) \exp (\alpha \lambda t_k) = \max \{ x_2 (t_k) \times \exp (- \lambda t_k) \} = - (2a + 2b) \alpha^\alpha, k \in \mathbb{Z} \), are fulfilled. From them we have \( \lambda [x_1] = \lambda [x_2] = \lambda_1 \). Let \( \lambda_2 (t) \) be equal to \( \exp (- \alpha t) \) on the segments \( [t_k, t_k + 1] \), \( k \in \mathbb{Z} \), and on all remaining intervals of the half-axis \( t \geq 0 \) let it be equal to \( \lambda_2 (t) \). Put all other elements of the matrix \( Q (t) \) to be zero. By the transformation \( y = X (t) \) we transform (1.4+Q) to (1.4) of the form (2) with \( \lambda (Q) < 0 \).

3. Let a solution \( y (t) = (y_1 (t), y_2 (t)) \) of (1.4+Q) realize the lower exponent \( \lambda_1 (A + Q) \).

4. We transform the constructed piecewise-constant functions \( a_2 (t) \) and \( \lambda_2 (t) \) into infinitely-differentiable ones \( \lambda_2 (t) \) and \( \lambda_2 (t) \) by means of replacing of them by the functions \( f (t); a_2 (t); a_2 (t) \) and \( f (t); \lambda_2 (t), \lambda_2 (t) \) on all intervals \( [t_k, t_k] \), one endpoint of which coincides with a point of discontinuity of these functions, of such a small length, that the systems (1.4) and (1.4+Q) are pairwise asymptotically equivalent (this is possible according to the Yu. S. Bogdanov–S. A. Maznikov theorem).
References


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