ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS, I

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In the present note, we consider the question of solvability of the boundary value problem

\[ u''(t) = F(u(t)), \quad u(a) = 0, \quad u(b) = 0, \]

where the continuous operator \( F : C^1([a, b]) \rightarrow L([a, b]) \) satisfies the Carathéodory conditions.

Before we proceed to formulate the basic results, let us introduce the following notation:

- \( R = ]-\infty, +\infty[ \), \( R_+ = [0, +\infty[ \);
- \( C([a, b]) \) is the space of continuous functions \( f : [a, b] \rightarrow R \) with the norm \( \|f\|_C = \max\{|f(t)| : a \leq t \leq b\} \);
- \( C^1([a, b]) \) is the space of continuously differentiable functions \( f : [a, b] \rightarrow R \) with the norm \( \|f\|_{C^1} = \|f\|_C + \|f'\|_C; C^1_0([a, b]) = \{f \in C^1([a, b]) : f(a) = 0, f(b) = 0\} \);
- \( C^1([a, b]) \) is the set of absolutely continuous, with its first derivative, functions \( f : [a, b] \rightarrow R \);
- \( L([a, b]) \) is the space of summable on \( [a, b] \) functions \( f : [a, b] \rightarrow R \) with the norm \( \|f\|_L = \int_a^b |f(s)| \, ds \);
- \( M(A, B) \) is the set of measurable functions \( F : A \rightarrow B \);
- \( K([a, b]) \) is the set of operators \( p : C^1([a, b]) \rightarrow M([a, b]; R) \);
- \( K((a, b) \times R, R_+) \) is the set of functions \( q : (a, b] \times R \rightarrow R_+ \) satisfying the Carathéodory condition;
- \( \sigma : L([a, b]) \rightarrow L([a, b]) \) is an operator defined by

\[ \sigma(p)(t) = \exp \left[ \int_a^t p(s) \, ds \right]. \]

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\[ \sigma : L([a, b]) \to L([a, b]) \] is an operator defined by
\[ \sigma(p)(t) = \frac{1}{\sigma(p)(t)} \int_{a}^{t} \sigma(p)(s) \, ds. \]

\[ [p(t)]_+ = \frac{1}{2} ([p(t)] + p(t)), \quad [p(t)]_- = \frac{1}{2} ([p(t)] - p(t)). \]

An operator \( l \in \mathcal{L}([a, b]) \) is said to be positive (negative) if for any nonnegative function \( u \in C([a, b]) \) the function \( l(u) \) is nonnegative (nonpositive).

In what follows, we assume \( F \in K([a, b]) \). Under solution of the equation (1) it is understood a function \( u \in \mathcal{C}([a, b]) \) which almost everywhere satisfies it.

**Definition.** Let \( l \in \mathcal{L}([a, b]) \). We say that a vector function \( (p, g_1, g_2) : [a, b] \to \mathbb{R}^3 \) belongs to the set \( V([a, b]; l) \) if \( p, g_1, g_2 \in L([a, b]) \) and for any function \( g \in M([a, b], R) \) satisfying
\[ g_1(t) \leq g(t) \leq g_2(t) \quad \text{for} \quad a < t < b, \]
there exists a positive function \( w \in \mathcal{C}([a, b]) \) such that
\[ w''(t) \leq p(t)w(t) + g(t)w'(t) + l(w)(t) \quad \text{for} \quad a < t < b. \]

**Remark.** Let \( l \in \mathcal{L}([a, b]) \) be a negative operator and \( p(t) + l(1)(t) \geq 0 \) for \( a < t < b \). Then for any \( g_1, g_2 \in L([a, b]) \) satisfying \( g_1(t) \leq g_2(t) \) for \( a < t < b \), we have \( (p, g_1, g_2) \in V([a, b]; l) \).

**Theorem 1.** Let on the set \( C_0^1([a, b]) \) the inequalities
\[ \begin{align*}
F(v)(t) - p_1(t)v(t) - p_2(v)(t)v'(t) - l(v)(t) &\geq -q(t, \|v\|_{C^1}), \\
g_1(t) &\leq p_2(v)(t) \leq g_2(t)
\end{align*} \tag{3} \]
be fulfilled, where \( l \in \mathcal{L}([a, b]) \) is a negative operator, \( p_2 \in K_0([a, b]) \), \( q \in K_1([a, b] \times \mathbb{R}, \mathbb{R}_+) \) is nondecreasing in the second argument and
\[ \lim_{x \to +\infty} \frac{1}{x} \int_{a}^{b} q(s, x) \, ds = 0. \tag{4} \]

Let, moreover,
\[ (p_1, g_1, g_2) \in V([a, b]; l). \]

Then the problem (1), (2) has at least one solution.

Mention two corollaries of Theorem 1 for the equation
\[ u''(t) = h(t)u(r(t)) + G(u)(t), \tag{5} \]
where \( G \in K([a, b]) \), \( r \in M([a, b], [a, b]) \), and \( h \in L([a, b]) \) is a nonpositive function.

**Corollary 1.** Let on the set \( C_0^1([a, b]) \) the inequality
\[ G(v)(t) \text{sgn } v(t) \geq -q(t, \|v\|_{C^1}) \tag{6} \]
be fulfilled, where \( q \in K_1([a, b] \times R, R_+) \) is nondecreasing in the second argument and satisfies (4). Moreover, let

\[
(b - \tau(t)) \int_a^b (s - a)|h(s)| \, ds + \\
+ (\tau(t) - a) \int_{\tau(t)}^c (h - \tau)|h(s)| \, ds < b - a \quad \text{for} \quad a < t < b.
\]

Then the problem (5), (2) has at least one solution.

**Corollary 2.** Let on the set \( C^0_g([a, b]) \) the inequality (6) be fulfilled, where \( q \in K_1([a, b] \times R, R_+) \) is nondecreasing in the second argument and satisfies (4). Let, moreover, there exist \( c \in [a, b] \) such that

\[
\int_a^c \sigma(p)(s)|h(s)| \, ds < 1, \quad \int_a^b \sigma_0(p)(s)|h(s)| \, ds < 1,
\]

\[
(t - \tau(t)) \int_a^c \frac{|h(s)|}{\sigma(p)(s)} \, ds \leq 1 \quad \text{for} \quad a < t < b,
\]

where \( p(t) > h(t)(\tau(t) - t) \) for \( a < t < b \). Then the problem (5), (2) has at least one solution.

Finally, we give a corollary of Theorem 1 for the equation

\[
u''(t) = p_1(t)u(t) + p_2(u(t)|u'(t) + h(t)u(\tau(t)) + G(u(t)), \quad (7)
\]

where \( p_2, G \in K([a, b]), \tau \in M([a, b], [a, b]), p_1, h \in L([a, b]) \) and \( h \) is positive.

**Corollary 3.** Let on the set \( C^0_g([a, b]) \) the inequalities (3) and (6) be fulfilled, where \( g_1, g_2 \in L([a, b]), q \in K_1([a, b] \times R, R_+) \) is nondecreasing in the second argument and satisfies (4). Let, moreover, there exist \( \lambda_i \in [0, 1], \alpha_{ij} \in [0, +\infty], i, j = 1, 2, \) and \( c \in [a, b] \) such that

\[
\int_0^{+\infty} ds \frac{s}{\alpha_{11} + \alpha_{12} s + s^2} > \frac{(c - a)^{1 - \lambda_1}}{1 - \lambda_1}, \quad \int_0^{+\infty} ds \frac{s}{\alpha_{21} + \alpha_{22} s + s^2} > \frac{(b - c)^{1 - \lambda_2}}{1 - \lambda_2}
\]

and

\[
(t - a)^{2\lambda_1} [p_1(t) + h(t)] \geq -\alpha_{11}, \quad (t - a)^{3\lambda_1} [g_1(t) + \frac{\lambda_3}{t - a} + (\tau(t) - t)h(t)] \geq -\alpha_{12} \quad \text{for} \quad a < t < c,
\]

\[
(b - t)^{2\lambda_2} [p_1(t) + h(t)] \geq -\alpha_{21}, \quad (b - t)^{3\lambda_2} [g_2(t) - \frac{\lambda_3}{b - t} + (\tau(t) - t)h(t)] \leq \alpha_{22} \quad \text{for} \quad c < t < b.
\]

Then the problem (7), (2) has at least one solution.
References


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