ON SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL AND INTEGRAL INEQUALITIES

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In the present note, we consider the questions of estimates of the solutions of the system of differential inequalities

$$dx(t)\cdot \text{sign}(t-t_0) \leq dC(t) \cdot x(t) + dq(t) \quad \text{for} \quad t \in [a, b] \setminus \{t_0\},$$

satisfying the condition

$$x(t_0) + (-1)^j d_j x(t_0) \leq c_0 + d_j C(t_0) \cdot c_0 + d_j q(t_0) \quad (j = 1, 2),$$

and of the solutions of the system of integral inequalities

$$x(t) \leq c_0 + \left( \int_{t_0}^{t} dC(\tau) \cdot x(\tau) + q(\tau) - q(t_0) \right) \text{sign}(t-t_0) \quad \text{for} \quad t \in [a, b],$$

satisfying the condition (2), where $t_0 \in [a, b]; c_0 \in \mathbb{R}^n; q \in \text{BV}(\mathbb{R}^n)$ and $C = (c_{ik})_{i,k=1}^{n,n}$ is the set of all real $n \times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$. If $X \in \mathbb{R}^{n \times n}$, then $\det(X)$ is the determinant of $X$, $I_n$ is the identity $n \times n$-matrix; $R^2 = \mathbb{R}^{n \times 1}$ is the set of all real column vectors $x = (x_i)_{i=1}^{n}$.

The following notation and definitions will be used: $\mathbb{R} = (-\infty, +\infty]$ is a closed segment, $\mathbb{R}^{n \times m}$ is the set of all real $n \times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$. If $X \in \mathbb{R}^{n \times m}$, then $\det(X)$ is the determinant of $X$, $I_n$ is the identity $n \times n$-matrix; $R^2 = \mathbb{R}^{n \times 1}$ is the set of all real column vectors $x = (x_i)_{i=1}^{n}$.

$\text{BV}(\mathbb{R}^n, \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,n} : [a, b] \to \mathbb{R}^{n \times m}$ such that every its component $x_{ik}$ has bounded total variation on $[a, b]$. If $I \subset \mathbb{R}$ is an interval, then $\text{BV}(I, \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n \times m}$ such that $X \in \text{BV}(\mathbb{R}^n, \mathbb{R}^{n \times m})$ for every $c, d \in I$. $X(t) = (x_{ik}(t))_{i,k=1}^{n,m}$ are the left and the right limits of $X$ at the point $t \in [a, b]$ ($X(a) - X(a)$), $X(b) = X(\hat{b})$, $d_1 X(t) = X(t) - X(t-), d_2 X(t) = X(t+) - X(t)$.

If $g : [a, b] \to R$ is a nondecreasing function, $x : [a, b] \to R$ and $a \leq s < t \leq b$, then

$$\int_{s}^{t} x(\tau) \, dg(\tau) = \int_{[s,t]} x(\tau) \, dg(\tau) + x(s) \, d_1 g(t) + x(t) \, d_2 g(s),$$

where $\int_{[s,t]} x(\tau) \, dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $[a, t]$ with respect to the measure $\mu_\alpha$ corresponding to the function $g$ (if $s = t$, then $\int_{s}^{t} x(\tau) \, dg(\tau) = 0$).

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If \( g_j : [a, b] \to R \) \((j = 1, 2)\) are nondecreasing functions, \( g = g_1 - g_2 \) and \( x : [a, b] \to R \), then

\[
\int_a^t x(\tau) dg(\tau) = \int_a^t x(\tau) dg_1(\tau) - \int_a^t x(\tau) dg_2(\tau) \quad \text{for} \quad a \leq s \leq t \leq b.
\]

If \( G = (g_{ik})_{n,k=1}^n \in \text{BV}([a, b], R^{n \times n}) \), \( x = (x_k)_{k=1}^n \in \text{BV}([a, b], R^n) \), then

\[
\int_a^t dG(\tau) \cdot x(\tau) = \left( \sum_{i=1}^n \int_a^t x_i(\tau) dg_{ik}(\tau) \right)^n \quad \text{for} \quad a \leq s \leq t \leq b.
\]

Let \( I \subset [a, b] \) be an interval and \( A \in \text{BV}(I, R^{n \times n}) \). A vector-function is said to be a solution of the system of the linear generalized ordinary differential equations \( dx(t) = dA(t) \cdot x(t) + dq(t) \) (inequalities \( dx(t) \leq dA(t) \cdot x(t) + dq(t) \)) on \( I \) if

\[
x(t) - x(s) - \int_s^t dA(\tau) \cdot x(\tau) - q(\tau) + q(s) = 0 \quad (\leq 0) \quad \text{for} \quad s \leq t \leq (s, t \in I).
\]

**Theorem 1.** Let \( c_{ik} \) \((i \neq k; i, k = 1, \ldots, n)\) be functions nondecreasing on \([a, b]\), \( C(t) = (c_{ik}(t))_{i,k=1}^n \).

\[
\det \left( t_i + (-1)^j d_j C(t) \right) \neq 0 \quad \text{for} \quad (-1)^j (t - t_0) \geq 0 \quad (j = 1, 2),
\]

\[
1 + d_j c_{ik}(t) > 0 \quad \text{for} \quad (-1)^j (t - t_0) \geq 0 \quad (j = 1, 2; \; i = 1, \ldots, n),
\]

and

\[
\sum_{i=1}^n d_j c_{ik}(t) < 1 \quad \text{for} \quad (-1)^j (t - t_0) < 0 \quad (j = 1, 2; \; k = 1, \ldots, n).
\]

Let, moreover, \( x \in \text{BV}([a, t_0], R^n) \cap \text{BV}([t_0, b], R^n) \) be a solution of the system (1) satisfying the condition (2). Then

\[
x(t) \leq y(t) \quad \text{for} \quad t \in [a, b] \setminus \{t_0\},
\]

where \( y \in \text{BV}([a, b], R^n) \) is a solution of the problem

\[
dy(t) = \left[ dC(t) \cdot y(t) + dq(t) \right] \text{sign}(t - t_0) \quad \text{for} \quad t \in [a, b] \setminus \{t_0\},
\]

\[
(-1)^j d_j y(t_0) - d_j C(t_0) \cdot y(t_0) + d_j q(t_0) \quad (j = 1, 2),
\]

\[
y(t_0) = c_0.
\]

**Theorem 2.** Let \( c_{ik} \) \((i, k = 1, \ldots, n)\) be functions nondecreasing on \([a, b]\) and (4) and (6) hold where \( C(t) = (c_{ik}(t))_{i,k=1}^n \). Then for every solution \( x \in \text{BV}([a, t_0], R^n) \cap \text{BV}([t_0, b], R^n) \) of the system (3) satisfying the condition (2), the estimate (7) holds, where \( y_0 \in \text{BV}([a, b], R^n) \) is a solution of the problem (8)-(10).

**Remark.** The condition

\[
\max_{k=1,\ldots,n} \sum_{i=1}^n |d_j c_{ik}(t)| < 1 \quad \text{for} \quad t \in [a, b] \quad (j = 1, 2)
\]

is not necessary.
guarantees the conditions (4)–(6). Moreover, in view of (4) the problem (8)–(10) has a unique solution (see [1, Theorem III.1.4]).

References


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