ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS, II

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In the present note, we continue the investigation of the question of solvability of the boundary value problem

\[ u''(t) = F(u(t)), \quad u(a) = 0, \quad u(b) = 0, \]

which was begun in [1]. Notation introduced there remains valid. Moreover, we assume that:

- \( L_2([a, b]) \) is the space of quadratically summable functions \( p : [a, b] \to \mathbb{R} \) with the norm \( \|p\|_{L_2} = \int_a^b p^2(s) \, ds \);
- \( \sigma_{ab} : L([a, b]) \to L([a, b]) \) is an operator defined by

\[
\sigma_{ab}(p)(t) = \frac{1}{\sigma(p)(t)} \int_a^t \sigma(p)(s) \, ds \cdot \int_t^b \sigma(p)(s) \, ds.
\]

**Theorem 1.** Let on the set \( C_0([a, b]) \) the inequalities

\[
\int_a^b \frac{[F(v)(s) - g(s)v'(s)] - g(s)v'(s)}{\sigma(g)(s)} \, ds \leq \alpha_0 \left( \left\| \frac{v'}{\sqrt{\sigma(g)}} \right\|_{L_2}^2 \right) + \alpha_1 \left( \left\| v \right\|_{C^1} \cdot \beta_1 \left( \left\| \frac{v'}{\sqrt{\sigma(g)}} \right\|_{L_2}^2 \right) \right),
\]

\[
\left\| (F(v) - g v') \cdot \text{sgn } v \right\|_{L_2} \leq \alpha_2 \left( \left\| v \right\|_{C^1} \right) + \beta_2 \left( \left\| v' \right\|_{L_2}^2 \right)
\]

be fulfilled, where \( g \in L([a, b]), \gamma + \delta = 1 \), and continuous nondecreasing functions \( \alpha_i, \beta_j : R_+ \to R_+ \), \( i = 0, 2, j = 1, 2 \), satisfy the conditions

\[
\limsup_{x \to +\infty} \frac{1}{x} \alpha_0(x) < 1,
\]

\[
\lim_{x \to +\infty} \frac{1}{x} \alpha_2(x) = 0, \quad \beta_1(x) = O(x), \quad i = 1, 2.
\]

Then the problem (1), (2) has at least one solution.

**Corollary 1.** Let on the set \( C_0'([a, b]) \) the inequality

\[
\left[ F(v)(t) - p(t)v(t) - g(t)v'(t) - l(v)(t) \right] \cdot \text{sgn } v(t) \geq -q \left( \|v\|_{C^1} \right)
\]

be fulfilled, where \( q \in L([a, b]), \gamma + \delta = 1 \) and \( \alpha_i, \beta_j : R_+ \to R_+ \), \( i = 0, 2, j = 1, 2 \), satisfy the conditions

\[
\limsup_{x \to +\infty} \frac{1}{x} \alpha_0(x) < 1,
\]

\[
\lim_{x \to +\infty} \frac{1}{x} \alpha_2(x) = 0, \quad \beta_1(x) = O(x), \quad i = 1, 2.
\]

Then the problem (1), (2) has at least one solution.

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be fulfilled, where $t \in \mathcal{L}([a, b])$, $p, q \in L([a, b])$, $q \in K_1([a, b] \times R, R_+)$ is nondecreasing in the second argument, and

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s, x) \, ds = 0. \tag{4}$$

Moreover, let

$$\frac{1}{\int_a^b \sigma(g)(s) \, ds} \cdot \int_a^b \sigma_{ab}(g)(s)p(s) \, ds + \frac{1}{2} \parallel \hat{l} \parallel < 1,$$

where $\hat{l} : L([a, b]) \to L([a, b])$ is the operator defined by the equality

$$\hat{l}(v)(t) = \frac{1}{\sqrt{\sigma(g)(t)}} \sqrt{\sigma_{ab}(g)(t)} l(v)(t).$$

Then the problem (1), (2) has at least one solution.

Mention two corollaries of Theorem 1 for the equation

$$u''(t) = p(t)u(t) + q(t)u'(t) + h(t)(u(\tau(t)) + G(u(t), \tau(t))), \tag{5}$$

where $p, q, h \in L([a, b])$, $\tau \in M([a, b] \times [a, b])$ and $G \in K([a, b])$.

**Corollary 2.** Let on the set $C^0([a, b])$ the inequality

$$G(v)(t) \sign v(t) \geq -q(t, \|v\|_{C^0}) \tag{6}$$

be fulfilled, where $q \in K_1([a, b] \times R, R_+)$ is nondecreasing in the second argument and satisfies (4). Let, moreover,

$$\frac{1}{\int_a^b \sigma(g)(s) \, ds} \cdot \int_a^b \sigma_{ab}(g)(s)p(s) + h(s) \, ds +$$

$$+ \frac{1}{\int_a^b \sigma(g)(s) \, ds} \cdot \int_a^b \sqrt{\sigma_{ab}(g)(s)} \sqrt{\frac{\int_a^b \sigma(g)(s) \, ds}{\sigma(g)(s)}} |h(s)| \, ds < 1. \tag{7}$$

Then the problem (5), (2) has at least one solution.

**Remark 1.** Note that, unlike Corollaries 1–3 in [1], the restriction imposed on the sign of the function $h$ is not required here.

In the case where

$$h(t) \geq 0 \quad a < t < b, \tag{8}$$

the condition (7) can be somewhat improved. More exactly, we have

**Corollary 3.** Let on the set $C^0([a, b])$ the inequality (6) be fulfilled, where $q \in K_1([a, b] \times R, R_+)$ is nondecreasing in the second argument and satisfies (4). Let, moreover, the inequality (8) holds, and

$$\frac{1}{\int_a^b \sigma(g)(s) \, ds} \cdot \int_a^b \sigma_{ab}(g)(s)[p(s)]_+ \, ds + \frac{1}{\int_a^b \sigma(g)(s) \, ds} \int_a^b \frac{h(s)}{\sigma(g)(s)} \int_s^b \sigma(g)(\xi) \, d\xi \, ds < 1.$$
Then the problem (5), (2) has at least one solution.

**Theorem 2.** Let on the set \( C^2_0([a, b]) \) the inequality (3) be fulfilled, where \( l \in L([a, b]) \) is a positive operator, \( p, g \in L([a, b]) \), and \( q \in K([a, b] \times R, R_+) \) is nondecreasing in the second argument and satisfies (4). Let, moreover,

\[
\int_a^b \sigma_{ab}(g)(s) |p(s)| \, ds < \int_a^b \sigma(g)(s) \, ds
\]

and

\[
\int_a^b \frac{l(1)(s)}{\sigma(g)(s)} \, ds \leq \frac{16}{\int_a^b \sigma(g)(s) \, ds} \left( 1 - \frac{1}{\int_a^b \sigma(g)(s) \, ds} \cdot \int_a^b \sigma_{ab}(g)(s) |p(s)| \, ds \right).
\]

Then the problem (1), (2) has at least one solution.

**References**


Authors' addresses:

A. Lomtatidze
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akari St., Tbilisi 380093
Georgia

S. Mukhigulashvili
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 390093
Georgia