

CHARACTERIZATION OF SEMIENTIRE GRAPHS WITH  
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*Abstract.* The purpose of this paper is to give characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we establish necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number 2.

*Keywords:* semientire graph, vertex-semientire graph, edge-semientire graph, crossing number, forbidden subgraph, homeomorphic graphs

*MSC 2000:* 05C50, 05C99

## 1. INTRODUCTION

Graphs considered here are simple graphs (without loops and multiple edges). A graph is said to be embedded in a surface when it is drawn on  $S$  so that no two edges intersect. A graph is planar if it can be embedded in the plane. By a plane graph we mean a graph embedded in the plane as opposed to a planar graph.

If there exists an edge  $e_1 = uv$  in a plane graph  $G$ , we say that the vertices  $u, v$  are adjacent to each other and both incident to the edge  $e_1 = uv$ . The edge  $e_1 = uv$  is said to be adjacent to an edge  $e_2$  if and only if  $e_2 = uw$  or  $e_2 = vw$ , where  $w$  is a vertex of  $G$  distinct from  $u$  and  $v$ . A region of  $G$  is adjacent to the vertices and edges which are on its boundary, and two regions of  $G$  are adjacent if their boundaries share a common edge. In this paper, vertices, edges and regions are called the elements of  $G$ .

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Kulli and Akka [2] introduced the concepts of a vertex-semientire graph and an edge-semientire graph of a graph. The vertex-semientire graph  $e_v(G)$  of a plane graph  $G$  is the graph whose vertex set is the union of the vertex set and the region set of  $G$  and in which two vertices are adjacent if and only if the corresponding elements (two vertices, two regions or a vertex and a region) of  $G$  are adjacent. The edge-semientire graph  $e_e(G)$  of a plane graph  $G$  is the graph whose vertex set is the union of the edge set and the region set of  $G$  and in which two vertices are adjacent if and only if the corresponding elements (two edges, two regions or an edge and a region) of  $G$  are adjacent. For other definitions see [1].

In [2], Kulli and Akka established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs are planar and outerplanar. Further, in [3], Kulli and Muddebihal established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number one. In addition, they established necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number one.

The main results of this paper are characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we give characterizations in terms of forbidden subgraphs of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2.

The following will be useful for proving our theorems.

**Theorem A** [2]. *Let  $G$  be a connected plane graph. Then  $e_v(G)$  is planar if and only if  $G$  is a tree.*

**Theorem B** [2]. *Let  $G$  be a connected plane graph. Then  $e_e(G)$  is planar if and only if  $\Delta(G) \leq 3$  and  $G$  is a tree.*

**Theorem C** [3]. *Let  $G$  be a connected plane graph. Then  $e_v(G)$  has crossing number 1 if and only if  $G$  is unicyclic.*

**Theorem D** [3]. *The edge-semientire graph  $e_e(G)$  of a connected plane graph  $G$  has crossing number 1 if and only if (1) or (2) holds.*

- (1)  $\Delta(G) = 3$ ,  $G$  is unicyclic and such that at least one vertex of degree 2 is on the cycle.
- (2)  $\Delta(G) = 4$ ,  $G$  is a tree and has exactly one vertex of degree 4.

## 2. MAIN RESULTS

In the next theorem, we present a characterization of graphs whose vertex-semientire graphs have crossing number 2.

**Theorem 1.** *Let  $G$  be a connected plane graph. Then  $e_v(G)$  has crossing number 2 if and only if  $G$  has exactly two cycles and these cycles are its blocks.*

*Proof.* Suppose  $e_v(G)$  has crossing number 2. Assume that  $G$  is a tree. Then by Theorem A,  $e_v(G)$  is planar, a contradiction.

Assume that  $G$  has at least three cycles. Suppose each cycle is a block of  $G$ . Then by Theorem C, each block which is a cycle in  $G$  gives at least one crossing in  $e_v(G)$ . Hence  $e_v(G)$  has at least three crossings, a contradiction. Thus  $G$  has exactly two cycles.

Suppose two cycles lie in a block. Then  $G$  has a subgraph homeomorphic to  $K_4 - x$ .  $G$  has two interior regions  $r_1$  and  $r_2$  and the exterior region  $R$ . In  $e_v(G)$ , the vertices  $r_1$ ,  $r_2$  and  $R$  are mutually adjacent, since the regions  $r_1$ ,  $r_2$  and  $R$  are mutually adjacent in  $G$ . Then in each adjacency there exists at least one crossing. Hence  $e_v(G)$  has at least 3 crossings, a contradiction. Thus we conclude that  $G$  has exactly two cycles as blocks.

Conversely, assume that  $G$  has exactly two cycles  $C_i$ ,  $i = 1, 2$ , which are both blocks. Also, let each edge which is not on  $C_i$  be a block of  $G$ . Let  $r_i$ ,  $i = 1, 2$  be two interior regions of  $C_i$  and  $R$  the exterior region of  $G$ . In  $e_v(G)$ , the vertex  $r_i$  is adjacent to each vertex of  $C_i$  without crossings, the vertex  $R$  is adjacent to each vertex of  $G$  without crossings and the vertex  $R$  is adjacent to  $r_i$  with two crossings.

Thus  $e_v(G)$  has crossing number 2. This completes the proof of the theorem.

In the next theorem, we obtain a characterization of graphs whose edge-semientire graphs have crossing number 2. □

**Theorem 2.** *The edge-semientire graph  $e_e(G)$  of a connected plane graph  $G$  has crossing number 2 if and only if*

- 1)  $\deg v \leq 4$  for every vertex  $v$  of  $G$ , and  $G$  is a tree and has exactly two vertices of degree 4, or  $G$  is not a tree and has exactly one cutvertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle
- or
- 2)  $\deg v \leq 3$  for every vertex  $v$  of  $G$  and  $G$  has exactly two cycles and these cycles are its blocks in which at least one vertex of degree 2 lies on each cycle, or  $G$  is unicyclic and such that no vertex of degree 2 is on the cycle.

Proof. Suppose the edge-semientire graph  $e_e(G)$  of a connected plane graph  $G$  has crossing number 2. Then it is nonplanar. By Theorem B or D,  $G$  is a tree with  $\Delta(G) \geq 4$  or  $G$  is not a tree and  $\Delta(G) \leq 3$ .

Suppose  $G$  is a tree with  $\deg v \geq 4$  for some vertex  $v$  of  $G$ . We consider the following cases.

*Case 1.* Suppose  $\deg v \geq 5$  for some vertex  $v$  of the tree  $G$ . Then clearly  $c(e_e(G)) > 2$ , a contradiction. Hence  $\Delta(G) \leq 4$ .

*Case 2.* Suppose  $\deg v = 4$  for some vertex  $v$  of  $G$ . Assume  $G$  has at least 3 vertices of degree 4. Then  $L(G)$  has at least 3 subgraphs isomorphic  $K_4$ . By the definition of  $e_e(G)$ ,  $L(G)$  is a subgraph of  $e_e(G)$ . The vertex  $R$  in  $e_e(G)$  which corresponds to the exterior region is adjacent to every vertex of  $L(G)$ , which gives at least 3 subgraphs isomorphic  $K_5$  in  $e_e(G)$ . Hence  $c(e_e(G)) > 2$ , a contradiction. Thus  $G$  has at most two vertices of degree 4.

Suppose  $G$  is not a tree and assume  $\deg v = 4$  for some vertex  $v$  of  $G$ . We consider 2 cases.

*Case 1.* Assume  $G$  has at least two vertices of degree 4 and at least one cycle  $C$ . Then  $L(G)$  has at least 2 subgraphs isomorphic to  $K_4$  and at least one subgraph  $L(C)$ . By the definition of  $e_e(G)$ ,  $L(G) \subset e_e(G)$ . The vertex  $r$  in  $e_e(G)$  (which corresponds to an interior region of  $C$ ) is adjacent to every vertex of  $L(C)$ . This gives one wheel  $W$ . The vertex  $R$  in  $e_e(G)$  is adjacent to every vertex of two  $K_4$  and  $W$  of  $L(G)$ . This gives at least 3 subgraphs isomorphic to  $K_5$  in  $e_e(G)$ . Thus  $c(e_e(G)) \geq 3$ , a contradiction.

*Case 2.* Assume  $G$  has at least one vertex of degree 4, at least two cycles  $C_i$ ,  $i = 1, 2$  as blocks and let  $r_i$  be the interior regions of  $C_i$ . Then  $L(G)$  has at least one subgraph isomorphic to  $K_4$  and at least two subgraphs  $L(C_i)$ . In  $e_e(G)$ ,  $r_i$  is adjacent to every vertex of  $L(C_i)$ , which gives a wheel  $W_i$ . Since  $L(G) \subset e_e(G)$ , the vertex  $R$  in  $e_e(G)$  which corresponds to the exterior region is adjacent to every vertex of  $L(G)$  and  $r_i$ . This gives at least 3 subgraphs isomorphic to  $K_5$  in  $e_e(G)$ . Hence  $c(e_e(G)) > 2$ , a contradiction.

From cases 1 and 2 we conclude that  $G$  has exactly one vertex of degree 4 and exactly one cycle.

Suppose  $G$  has exactly one vertex  $v$  of degree 4 and a cycle  $C$ . Assume that every vertex of  $C$  has degree at least three. Let  $e_i$ ,  $i = 1, 2, 3$  and 4 be edges adjacent to  $v$ . Then  $L(G)$  has exactly one subgraph isomorphic to  $K_4$  and exactly one cycle  $L(C)$ . Let  $r$  be the interior region of  $C$  and  $R$  the exterior region of  $G$ . In  $e_e(G)$ , the vertex  $r$  is adjacent to every vertex of  $L(C)$  without crossing, which gives  $e_e(G) - R$ . We get two wheels  $L(C) + r$  and  $K_3 + e_i (= K_4)$ ,  $i = 1, 2, 3$  or 4 in  $e_e(G) - R$ . In  $e_e(G) - \{rR, Re_i\}$ , the vertex  $R$  is adjacent to every vertex of  $e_e(G) - \{r, e_i\}$  without crossings. In  $e_e(G)$  it is easy to see that the edges  $Re_i$  and  $rR$  cross respectively at

least one edge and at least 2 edges of  $e_e(G) - \{rR, re_i\}$ . Thus  $e_e(G)$  has at least 3 crossings, a contradiction. This proves (1).

Assume  $G$  is not a tree and  $\deg v \leq 3$  for every vertex  $v$  of  $G$ . We consider three cases.

*Case 1.* Assume  $G$  has at least 3 cycles. Suppose each cycle has at least one vertex of degree two and each cycle is a block of  $G$ . Let  $R$  and  $r_i$ ,  $i = 1, 2, 3$  be vertices in  $e_e(G)$  which correspond to the exterior and interior regions of  $G$ . Then  $e_e(G) - R$  has at least 3 blocks each of which is a wheel. In  $e_e(G)$ ,  $R$  is adjacent to each wheel. We get at least 1 crossing in each case. It is clear that  $e_e(G)$  has at least 3 crossings, a contradiction.

*Case 2.* Suppose  $G$  has at least two cycles in a block. Then  $G$  has a subgraph homeomorphic to  $K_4 - x$ . Obviously  $G$  has 2 interior regions, say  $r_1$  and  $r_2$ , and the exterior region  $R$ . Clearly  $e_e(G) - R$  has a block in which the edge joining the vertices  $r_1$  and  $r_2$  has two crossings. Also in  $e_e(G)$ , the vertex  $R$  is adjacent to  $r_1$  and  $r_2$ , which makes two more crossings. Thus  $c(e_e(G)) \geq 4$ , a contradiction.

From the above cases, we conclude that  $G$  has at most two cycles  $C_i$  as blocks.

Assume  $G$  has no vertex of degree 2 on each cycle  $C_i$ . The interior regions  $r_1$  and  $r_2$  are adjacent respectively to every vertex of  $C_1$  and  $C_2$  without crossings and this gives  $e_e(G) - R$  where  $R$  is the exterior region. The vertex  $R$  is adjacent to each vertex of  $e_e(G) - \{r_1, r_2\}$  without crossings. In  $e_e(G)$ ,  $r_1R$  and  $r_2R$  are edges. Clearly each  $r_iR$  crosses at least 2 edges in  $e_e(G) - \{r_1R, r_2R\}$ . Thus  $c(e_e(G)) \geq 4$ , a contradiction.

Suppose  $G$  is unicyclic and all vertices of the cycle  $C$  are of degree less than 3. Assume that at least one vertex of the cycle  $C$  of  $G$  has degree 2. Then by condition (1) of Theorem D,  $e_e(G)$  has exactly one crossing, a contradiction. This proves (2).

Conversely, suppose  $G$  is a graph satisfying conditions (1) or (2). Then by Theorem B or D,  $e_e(G)$  has crossing number at least 2. We now show that its crossing number is at most 2. Assume first that  $G$  satisfies condition (1). We consider 3 cases.

*Case 1.* Suppose  $G$  is a tree and has exactly two vertices of degree 4. Then clearly  $e_e(G)$  has exactly two subgraphs, each isomorphic to  $K_5$ , and hence  $e_e(G)$  can be drawn with exactly two crossings.

*Case 2.* Suppose  $G$  is not a tree and has exactly one vertex of degree 4 and exactly one cycle  $C$  such that at least one vertex of degree 2 is on the cycle. Then it is easy to see that  $e_e(G)$  has exactly two crossings.

Now assume (2). Then  $G$  has exactly two cycles  $C_i$  as blocks in which at least one vertex of degree 2 lies on each cycle. Let  $r_i$ ,  $i = 1, 2$  be the interior regions of two circles  $C_i$  of  $G$ . The vertex  $r_i$  is adjacent to every vertex of  $L(C_i)$  without crossings, which gives  $e_e(G) - R$  where  $R$  is the exterior region of  $G$ . Obviously  $e_e(G) - R$

has at least two blocks each of which is a wheel with at least one boundary edge. In  $e_e(G) - \{r_1R, r_2R\}$  the vertex  $R$  is adjacent to every vertex of  $e_e(G) - \{r_1, r_2\}$  without crossings. By the definition of  $e_e(G)$ ,  $r_1R$  and  $r_2R$  are edges. Hence either of  $r_1R$  and  $r_2R$  crosses exactly one edge of  $e_e(G) - \{r_1R, r_2R\}$  and gives  $e_e(G)$ . Hence  $e_e(G)$  has exactly two crossings.

Suppose  $G$  is unicyclic in which no vertex of degree 2 is on the cycle  $C$ . Let the vertices  $r$  and  $R$  correspond to the interior and exterior regions of  $G$ , respectively. The vertex  $r$  is adjacent to every vertex of  $L(C)$  and gives one wheel together with a triangle on each side (in  $e_e(G) - R$ ) without crossings. In  $e_e(G) - rR$ , the vertex  $R$  is adjacent to every vertex of  $e_e(G) - r$  without crossings. Thus the edge  $rR$  crosses exactly two boundary edges of  $e_e(G) - rR$  and gives  $e_e(G)$ . Hence  $c(e_e(G)) = 2$ . This completes the proof of the theorem.  $\square$

### 3. FORBIDDEN SUBGRAPHS

With help of Theorems 1 and 2 we now characterize graphs whose semientire graphs have crossing number 2, in terms of forbidden subgraphs.

**Theorem 3.** *Suppose a connected plane graph  $G$  has at least two cycles as blocks. The vertex-semientire graph  $e_v(G)$  has crossing number 2 if and only if it has no subgraph homeomorphic to  $G_i$ ,  $i = 12, 13, 14, 16, \dots, 19$  or 20 (Fig. 1).*

*Proof.* Assume a connected plane graph  $G$  has at least two cycles. Suppose  $c(e_v(G)) = 2$ . Then by Theorem 1,  $G$  has at most two cycles as blocks. It follows that  $G$  has no subgraph homeomorphic to  $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$  or  $G_{20}$ .

Conversely, suppose  $G$  has at least two cycles as blocks and has no subgraph homeomorphic to  $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$  or  $G_{20}$ .

Suppose  $G$  has at least 3 cycles each of them being a block of  $G$ . Then  $G$  has a subgraph homeomorphic to  $G_{12}, G_{13}, G_{16}, G_{17}, G_{18}, G_{19}$  or  $G_{20}$ , a contradiction.

Suppose  $G$  has a block which contains at least two cycles. Then  $G$  has a subgraph homeomorphic to  $G_{14}$ , a contradiction.

In each case we have arrived at a contradiction. Thus Theorem 1 implies that  $c(e_v(G)) = 2$ . This completes proof.  $\square$

**Theorem 4.** *The edge-semientire graph  $e_e(G)$  of a connected plane graph  $G$  (with at least 5 vertices and 5 edges and  $\Delta(G) \leq 4$ ) has crossing number 2 if and only if  $G$  has no subgraph homeomorphic to  $G_i$ ,  $i = 1, 2, \dots, 14$  or 15 (Fig. 1).*

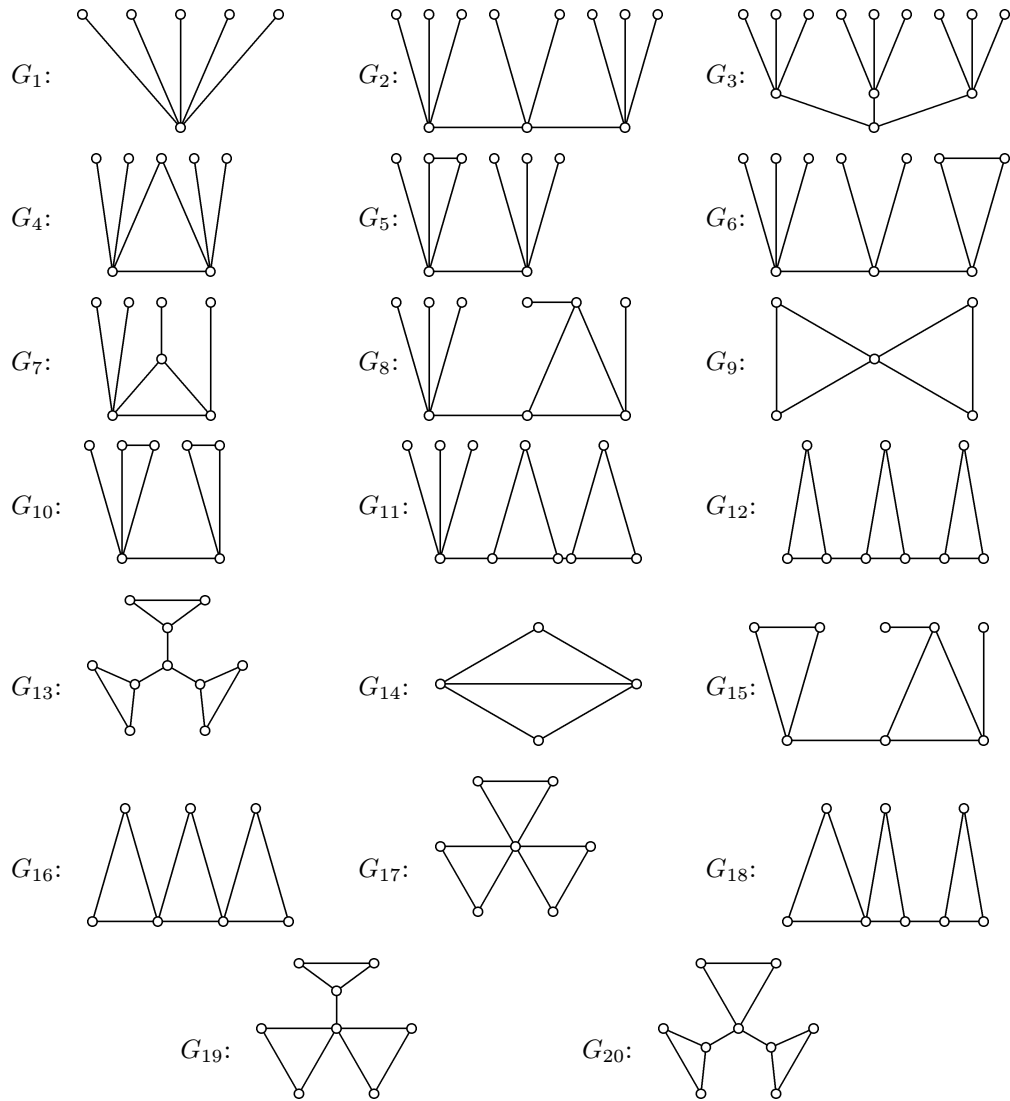


Fig. 1

PROOF. Assume  $G$  is a connected plane graph whose edge-semientire graph  $e_e(G)$  has crossing number 2. We prove that all graphs homeomorphic to  $G_i$ ,  $i = 1, 2, \dots, 14$  or 15 have  $c(e_e(G_i)) > 2$ . By Theorem 2, we have (1)  $\deg v \leq 4$  for every vertex  $v$  of  $G$  and  $G$  is a tree and has exactly two vertices of degree 4 or  $G$  is not a tree and has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle. Or (2)  $\deg v \leq 3$  for every vertex  $v$  of  $G$  and  $G$  has exactly two cycles as blocks in which at least one vertex of degree 2 is on each

cycle or  $G$  is unicyclic and such that no vertex of degree 2 is on the cycle. From (1) or (2) it follows that  $G$  has no subgraph homeomorphic to any one of the graphs  $G_i$ ,  $i = 1, 2, \dots, 15$ .

Conversely, assume that  $G$  is a connected plane graph and does not contain a subgraph homeomorphic to any one of the graphs  $G_i$ ,  $i = 1, \dots, 15$ . We shall show that  $G$  satisfies (1) or (2) and hence by Theorem 2,  $e_e(G)$  has crossing number 2. Suppose  $\deg v \geq 5$  for some vertex  $v$  of  $G$ . Then  $G$  contains a subgraph homeomorphic to  $G_1$ , a contradiction. Hence  $\deg v \leq 4$  for every vertex  $v$  of  $G$ . We consider the following two cases.

*Case 1.* Suppose  $G$  is a tree. Assume there exist at least three vertices of degree 4. Then  $G$  has a subgraph homeomorphic to  $G_2$  or  $G_3$ , a contradiction. Hence  $G$  has exactly two vertices of degree 4.

*Case 2.* Suppose  $G$  is not a tree. Then we consider two subcases.

*Subcase 2.1.* Suppose  $G$  is unicyclic  $C$ . Assume  $G$  has exactly two vertices  $v_1$  and  $v_2$  of degree 4. Then we consider 3 possibilities.

- a) If  $v_1, v_2 \in C$ , then  $G$  has a subgraph homeomorphic to  $G_4$ .
- b) If  $v_1$  or  $v_2 \in C$ , then  $G$  has a subgraph homeomorphic to  $G_5$ .
- c) If  $v_1, v_2 \notin C$ , then  $G$  has a subgraph homeomorphic to  $G_6$ .

In each case we have a contradiction. Thus  $G$  has exactly one vertex of degree 4 and exactly one cycle.

Suppose  $G$  has exactly one vertex  $v$  of degree 4 and exactly one cycle  $C$  such that no vertex of degree 2 is on the cycle. Then we consider two possibilities.

- a) If  $v \in C$ , then  $G$  has a subgraph homeomorphic to  $G_7$ , a contradiction.
- b) If  $v \notin C$ , then  $G$  has a subgraph homeomorphic to  $G_8$ , a contradiction.

Thus  $G$  has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle, or  $G$  is unicyclic with every vertex of degree 3 on the cycle.

*Subcase 2.2.* Assume  $G$  is not a unicyclic graph. Suppose  $G$  has exactly one vertex  $v$  of degree 4 and at least two cycles  $C_1$  and  $C_2$ , each of which has at least one vertex of degree 2. We consider the following three possibilities.

- a) If  $v \in C_1$  and  $C_2$ , then  $G$  has a subgraph homeomorphic to  $G_9$ .
- b) If  $v \in C_1$  or  $C_2$ , then  $G$  has a subgraph homeomorphic to  $G_{10}$ .
- c) If  $v \notin C_1$  and  $C_2$ , then  $G$  has a subgraph homeomorphic to  $G_{11}$ .

In each case we have a contradiction. Thus  $G$  has at least 2 cycles each of which has at least one vertex of degree 2. Assume  $\deg v \leq 3$  for every vertex  $v$  of  $G$ . Then we consider 3 cases.

*Case 1.* Suppose  $G$  has at least 3 cycles as blocks such that each block has at least one vertex of degree two. Then  $G$  has a subgraph homeomorphic to  $G_{12}$  or  $G_{13}$ , a contradiction.

*Case 2.* Suppose  $G$  has a block which contains at least two cycles. Then  $G$  has a subgraph homeomorphic to  $G_{14}$ , a contradiction.

Thus  $G$  has at most two cycles as blocks.

*Case 3.* Suppose  $G$  has exactly two cycles as blocks such that one block has no vertex of degree 2. Then  $G$  has a subgraph homeomorphic to  $G_{15}$ , a contradiction. Thus  $G$  has exactly two cycles such that each cycle has at least one vertex of degree 2, or  $G$  has exactly one cycle such that each vertex on the cycle is of degree 3.

We have exhausted all possibilities. In each case we found that  $G$  contains a subgraph homeomorphic to some of the forbidden subgraphs  $G_i$ ,  $i = 1, \dots, 15$ . Hence by Theorem 2,  $e_e(G)$  has crossing number 2. This completes the proof of the theorem.  $\square$

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