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On finite dimensional representations of non-connected reductive groups

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Abstract. We extend the classification of irreducible finite dimensional representations of almost simple algebraic groups over an algebraically closed field of characteristic zero to certain non-connected groups G where the component group is cyclic. We also extend some of Steinberg's results on the adjoint quotient $G \to T/W$ to these non-connected groups. These results are used to describe the geometry of θ-conjugacy classes of G^o, where θ is an automorphism of the connected group G^o. As an application, we show that there is a "functorial" correspondence between virtual (finite dimensional) characters of θ-invariant representations of G and virtual characters of an endoscopic group H of G.

Contents

1. Introduction

Let k denote an algebraically closed field of characteristic zero. Unless stated otherwise G will always denote a connected semisimple (linear algebraic) group

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over k. In this paper, we extend some results of Steinberg to non-connected groups. Let θ denote an (algebraic) automorphism of G of finite order which fixes a "splitting" of G (this will be defined later). Let $G \rtimes <\theta>$ denote the semi-direct product where the multiplication is given by $(g,\sigma)(g',\sigma') = (g\sigma(g'),\sigma\sigma'))$. G may be identified with a normal subgroup of $\overline{G}$.

In one of our main results, we describe a twisted analog of the "adjoint quotient $G \to T/W$", where T is a Cartan of G and W is its Weyl group, as well as prove some basic facts regarding the geometry of "twisted" conjugacy classes. As a corollary to this, we prove that a θ-conjugacy class of G is Zariski closed if it is θ-semisimple. As a related matter, we investigate the characters of the irreducible finite dimensional representations of non-connected semisimple groups of the form $G \rtimes <\theta>$. In another of our main results, we prove that there is a natural 1-1 correspondence between the irreducible finite dimensional representations of $G \rtimes <\theta>$ and those representations on a related "endoscopic" group (one must be careful in interpreting this statement - see Remark 3.3 below).

In general, if X denotes an algebraic variety defined over k then we often identify X with $X(k)$.

2. Background on conjugacy classes

Let G be a closed connected (algebraic) subgroup of $GL(V)$, where V is a finite dimensional k-vector space, and let θ be an algebraic automorphism of G of finite order. Denote the semidirect product described above by $\overline{G} = G \rtimes <\theta>$. Denote the connected components of $\overline{G}$ by $G = G.1, G.\theta, G.\theta^2, \ldots$. Without loss of generality (taking V to be the Lie algebra of G), we can (and do) assume that $\overline{G}$ is also a subgroup of $GL(V)$.

Lemma 2.1. *Let $\overline{G}$ be as above and let X denote a variety defined over k on which $\overline{G}$ acts. Then*

(a) every orbit is open in its closure,

(b) for each $x \in X$, the (Zariski) closure $\overline{G \cdot x}$ of an orbit $G \cdot x$ is a union of $G \cdot x$ and other orbits of smaller dimension,

(c) orbits of minimal dimension are closed.

For the proof in the connected case, see [10], §1.13.

Proof. Since θ acts on X, it is an isomorphism of X to itself. In particular, the set of orbits of $G.\theta^i$ on X is the same as the set of orbits of $G.1 \cong G$ on X. Therefore the Lemma above is a consequence of the connected case. ∎

Proposition 2.2. *(Jordan decomposition) Let $\overline{G}$ be as above. Each $g \in \overline{G}$ has a decomposition $g = su = us$, where $s \in \overline{G}$ is semisimple (as an element of $\overline{G}$ or, equivalently, as an endomorphism on V) and $u \in \overline{G}$ is unipotent (as an element of $\overline{G}$ or, equivalently, $u - 1$ is a nilpotent endomorphism on V). Furthermore, g determines s and u uniquely.*

For the proof, see [10], §§2.1-2.4. As a consequence, we have the following result.

Corollary 2.3. *Let G be as above. Let θ denote an automorphism of G of finite order. Then any $g.\theta \in \overline{G}$ has a decomposition*

$$g.\theta = (s.\theta)u = u(s.\theta),$$

for some unique $s.\theta$ semisimple in $\overline{G}$ and u unipotent. Equivalently,

$$g = s\theta(u) = us.$$

Definition 2.4. In the above corollary, we write $g_{\theta-ss} = s$ and $g_{\theta-un} = u$. We call g *θ-semisimple* if $g.\theta \in \overline{G}$ is semisimple (i.e., if $g = g_{\theta-ss}$). We call g *θ-unipotent* if $g_{\theta-ss} = 1$, so $g = \theta(u) = u$ for some unipotent u.

We call $x, y \in$G *θ-conjugate* if and only if $y = g^{-1}x\theta(g)$ for some $g \in G$.

Remark 2.5. The map

$$g \longmapsto g_{\theta-ss}$$

is not a morphism [8], §3.3. However, later we will construct a morphism (the twisted analog of the adjoint quotient) which is closely related to this map.

A *Borel pair* of a connected reductive group G is a pair (B, T), where T is a maximal torus contained in a Borel subgroup B of G. An automorphism of G which preserves a Borel pair (B, T) will be called *quasi-semisimple.* Let W denote the Weyl group of T and W^θ denote the θ-invariant elements of W.

Lemma 2.6. *Let G denote a connected reductive group over k. Assume that θ is a quasi-semisimple automorphism of G which is of finite order and fixes a Borel pair (B, T) of G. Then*

$$N_G(T.\theta)/T = W^\theta,$$

where Ws denote the Weyl group of T.

Proof. Write the Bruhat decomposition as $G = BWB$ (where we have temporarily identified W with a complete set of coset representatives of W in $N_G(T)$). The lemma follows from the claim: if $w \in W$ and $g = bwb' \in BwB$ satisfies $gT\theta(g)^{-1} = T$ then $g \in TwT$ and $w \in W^\theta$. We next prove this claim.

Let $B = NT = TN$, where N is the unipotent radical of B. Note that since θ preserves T and B, it preserves N. Suppose that $g = tnwn'$, for $n, n' \in N$ and $t \in T$. The equation $gT\theta(g)^{-1} = T$ implies $\theta(g)T\theta(g)^{-1} = \theta(g)g^{-1}T$. The theory of maximal tori implies $g\theta(g)^{-1} \in T$ and $\theta(g) \in N_G(T)$. In turn, this implies $g \in N_G(T)$ and that the image of $\theta(g)g^{-1}$ in W is the identity. This (by the Bruhat decomposition) implies the claim, from which the lemma follows. ■

For $x \in G$, let

$$\mathcal{C}_\theta(x) = \{g^{-1}x\theta(g) \mid g \in G\},$$

the *θ-conjugacy class of x*. A *splitting* of G is a triple $(B, T, \{X\})$, where (B, T) is a Borel pair of G and $\{X\}$ is a set of root vectors of T, one for each simple root of T in B. Let W and W^θ be as in the previous lemma. Let $T_\theta = T/(1-\theta)T$ denote the group of *coinvariants* of T.

The lemma below, based on the work of Steinberg [11], is Lemma 3.2.A in [4].

Lemma 2.7. *Let G denote a connected reductive group over k and assume that the automorphism θ is of finite order and fixes a splitting* [1] $(B, T, \{X\})$ *of* G*. If $s \in G$ is θ-semisimple then*

(a) each $\mathcal{O} = \mathcal{C}_\theta(s)$ meets T,

(b) the image of $\mathcal{O} \cap T$ in T_θ is a single W^θ-orbit.

Remark 2.8. This implies that there is a bijection [2]

$$G_{\theta-ss} \to T_\theta/W^\theta$$

(essentially the *abstract norm map* defined in in [4] §3.2) from the set of conjugacy classes of θ-semisimple elements of G to the quotient T_θ/W^θ. The construction of the above bijection should be compared with Corollary 5.4 below, which obtains this mapping from a character-theoretic perspective.

3. Some representation theory

Let G be a connected semisimple algebraic group over k. Let θ be an quasi-semisimple automorphism of G. Let $\overline{G}$ be as in the previous section.

If ρ is a finite dimensional representation of G, let

$$\rho^\theta(g) = \rho(\theta(g)), \qquad g \in G,$$

so ρ^θ is irreducible if and only if ρ is. We call a representation ρ *θ-invariant* if $\rho^\theta \cong \rho$. The purpose of this section is to classify such representations.

We denote the simple roots of (B, T) by Δ, the root system of (B, T) by $R = R(B, T)$, the root lattice of $T \subset G$ by $Q = \mathbb{Z}[\Delta] = \mathbb{Z}[R]$, the character lattice by $X = X^*(T)$, and the (abstract) weight lattice by $P = Q^\perp$ (the dual being taken with respect to the Cartan pairing), let P^+ denote the "cone" of dominant (abstract) weights, and let $\lambda_1, ..., \lambda_n \in P^+$ denote the fundamental (abstract) weights. We have

$$Q \subset X \subset P.$$

Let λ denote a dominant character and let ρ_λ denote the irreducible representation whose highest weight is λ. Let X^+ denote the "cone" of dominant characters. If $\rho = \rho_\lambda$, with $\lambda \in X^+$, let

$$\chi_\rho(g) = \chi_\lambda(g) = \mathrm{trace}(\rho_\lambda(g)), \quad g \in G. \tag{1}$$

Finally, let X^θ denote the subgroup of θ-invariant characters in X and let $X^{\theta+}$ denote the dominant ones. We call such characters *θ-dominant.*

It is clear that if $\lambda \in X^\theta$ then the restriction of λ to $(1-\theta)T$ is trivial and hence λ is well-defined as a character of T_θ. Thus, there is a natural map $\eta : X^*(T)^\theta \to X^*(T_\theta)$. The kernel of the map $t \longmapsto t/\theta(t)$, $T \to (1-\theta)T$, is

[1] As was pointed out by the referee, this condition is stronger than the notion of a quasi-central automorphism in the sense of [1]; Lemma 2.7 holds for quasi-central automorphisms but not Proposition 3.9 below.

[2] This is only a set-theoretic mapping since $G_{\theta-ss}$ is not in general a variety.

T^θ. Thus $T/T^\theta \cong (1-\theta)T$. If we knew that T^θ was connected (and hence $T, T^\theta, (1-\theta)T$ were all products of $k^\times$'s) then we could conclude $T_\theta \cong T^\theta$. This connectedness is proven in [11], §8. This also proves that each element $t \in T$ can be uniquely factored $t = t_1 t_2$, where $t_1 \in T^\theta$ and $t_2 \in (1-\theta)T$ (see also Corollaire 1.33 in [1]). From this fact, one can show that η is an isomorphism. These facts are summarized in the following lemma.

Lemma 3.1. *Let G, θ, T be as in Lemma 2.6. There is a natural isomorphism*

$$\eta : X^*(T)^\theta \to X^*(T_\theta).$$

Furthermore, there is a (non-canonical) isomorphism $T_\theta \cong T^\theta$. Indeed, each element $t \in T$ can be uniquely factored $t = t_1 t_2$, where $t_1 \in T^\theta$ and $t_2 \in (1-\theta)T$.

Proposition 3.2. *Let ρ be a finite dimensional representation of G. The following are equivalent:*

(a) ρ is θ-invariant,

(b) ρ extends to a finite dimensional representation of $\overline{G}$.

Remark 3.3. In the introduction, we stated that one aim of this paper is to provide a correspondence between irreducible representations of $\overline{G}$ and irreducible representations of an endoscopic group. This is not exactly what we shall do. In fact, our correspondence will between θ-invariant representations of G (which extend to $\overline{G}$) and irreducible representations of an endoscopic group. Be that as it may, the remaining irreducible representations of $\overline{G}$ can be provided with a similar correspondence (though with a different endoscopic group) thanks to the following well-known result.

Lemma 3.4.
- *If $\overline{\rho}$ denotes an irreducible finite dimensional representation of $\overline{G}$ then there is an irreducible representation ρ of G such that*
  - *$\rho^{\theta^m} \cong \rho$, for some $m|d$ and m is chosen as small as possible,*
  - *ρ extends to the non-connected group $G \rtimes < \theta^m >$,*
  - *$\overline{\rho} \cong Ind_{G\rtimes<\theta^m>}^{\overline{G}} \rho$.*
- *Let ρ be an irreducible finite dimensional representation of G and let θ be an automorphism of finite order d of G. If all d of the representations ρ^{θ^i}, $0 \le i \le d-1$, are inequivalent then $Ind_G^{\overline{G}} \rho$ is an irreducible representation of $\overline{G}$.*

Proof of the Lemma. This follows from a far more general result of Mackey [6] (see also Theorem 2, Ch III, §B of [5]). ■

Proof of the Proposition. Let (ρ, V_ρ) denote a finite dimensional representation of G. If $\rho^\theta \cong \rho$ then there is an $A \in GL(V_\rho)$ of finite order dividing d such that $\rho(\theta(g)) = A^{-1}\rho(g)A$, for all $g \in G$. Define

$$\overline{\rho}(g.\theta^i) = \rho(g)A^{-i}, \quad g \in G, \quad 0 \le i \le d.$$

where d denotes the order of θ. It may be verified that this defines an extension of ρ to $\overline{G}$, so (a) implies (b).

To prove the converse, (b) implies (a), let $\overline{\rho}$ denote an extension of ρ to $\overline{G}$, so the image of $\overline{G}$ under $\overline{\rho}$ is also a semi-direct product. We may write

$$\overline{\rho}(g.\theta^i) = \rho(g)A^{-i}, \quad g \in G,$$

for some $A \in GL(V)$. Calculating

$$\rho(g\theta(g'))A^{-1} = \overline{\rho}(g\theta(g').\theta) = \overline{\rho}(g.\theta)\rho(g') = \rho(g)A^{-1}\rho(g') = \rho(g)\rho(\theta(g'))A^{-1},$$

we find that $\rho(g)\rho(\theta(g'))A^{-1} = \rho(g)A^{-1}\rho(g')$. This implies (a). ■

The following lemma is, as we shall see, an immediate consequence of the Weyl character formula.

Lemma 3.5. *(a) If $\lambda \in X^{\theta+}$ then $\rho_\lambda \cong \rho_\lambda^\theta$. Conversely, if $\lambda \in X^+$ and $\rho_\lambda = \rho_\lambda^\theta$ then $\lambda \in X^{\theta+}$.*

(b) Let λ^θ denote the highest weight of ρ_λ^θ, $\rho_{\lambda^\theta} = \rho_\lambda^\theta$. Let $V = X \otimes \mathbb{R}$, so we may extend the action of θ on X to V by linearity. These two actions (the above action of θ on V and the action of θ on dominant weights $\theta : \lambda \longmapsto \lambda^\theta$) are compatible.

For V as in the above lemma, let V^θ denote the subspace of θ-invariants and, for any subset $S \subset V$, let $S^\theta = S \cap V^\theta$.

Proof. (a) This follows from the direct construction of a representation with highest weight λ, as the referee pointed out. However, we give the following simple, analytic proof.

We have

$$\chi_\lambda = \frac{A_{\lambda+\rho_0}}{A_{\rho_0}}, \tag{2}$$

where ρ_0 denotes half the sum of the positive roots of (B, T) and

$$A_\mu = \sum_{w \in W} (-1)^{sgn(w)} w\mu$$

is the alternating sum over the equivalence class [2], Theorem 24.2. Because of this, if $\lambda \in X^{\theta+}$ then

$$\chi_{\rho_\lambda} = \chi_{\rho_{\lambda^\theta}} = \chi_{\rho_\lambda^\theta}.$$

This implies $\rho_\lambda \cong \rho_\lambda^\theta$ since the character determines the equivalence class of the representation.

On the other hand, suppose $\rho_\lambda \cong \rho_\lambda^\theta$. The character formula above implies

$$\frac{A_{\lambda+\rho_0}(\theta(t))}{A_{\rho_0}(\theta(t))} = \frac{A_{\lambda+\rho_0}(t)}{A_{\rho_0}(t)},$$

for all $t \in T$. Since θ permutes the set of positive roots (it must since it preserves (B, T)), we have $\theta(\rho_0) = \rho_0$. Thus,

$$\frac{A_{\lambda+\rho_0}(\theta(t))}{A_{\rho_0}(\theta(t))} = \frac{A_{\theta(\lambda)+\rho_0}(t)}{A_{\rho_0}(t)},$$

which implies that $\rho_\lambda = \rho_\lambda^\theta$ has highest weight $\theta(\lambda)$. Since the highest weight is unique, it follows that $\lambda = \theta(\lambda) = \lambda^\theta$, as desired.

(b) Part (b) is a consequence of the above proof. ■

From this it follows that the irreducible, finite dimensional, θ-invariant representations are in 1-1 correspondence with the elements of $X^{\theta+}$.

For $w \in W$ and $\chi \in X$, define $(w\chi)(t) = \chi(w^{-1}tw)$ for all $t \in T$. Clearly, $w\chi \in X$. If $\chi, \chi' \in X$ then we define

$$\chi \sim \chi'$$

if there is an element of the Weyl group $w \in W$ such that $\chi' = w\chi$. This is an equivalence relation on X and the set of equivalence classes, i.e. W-orbits, will be denoted by X/W. The classes in X/W are in natural 1-1 correspondence with the set X^+ since each class in X/W contains a unique dominant highest character.

If $\chi, \chi' \in X^\theta$ then we define

$$\chi \sim_\theta \chi'$$

if there is an element $w \in W$ such that $\chi' = w\chi$ (we do not know if it suffices to assume $w \in W^\theta$ in this definition). This is an equivalence relation on X^θ.

Now define

$$Symm[\chi] = \oplus_{\chi' \sim \chi} \chi',$$

for any $\chi \in X$. Note $Symm[\chi] \cong Symm[\chi']$ if and only if $\chi \sim \chi'$. Let

$$Symm_\theta[\chi] = \oplus_{\chi' \sim_\theta \chi} \chi',$$

for $\chi \in X^\theta$.

The following well-known lemma describes how the restriction of a irreducible character to a Cartan subgroup decomposes. We will prove the "twisted analog" of this result.

Lemma 3.6. *(a) For $\lambda \in X$ dominant,*

$$\chi_\lambda|_T = \sum_{\substack{\mu \leq \lambda \\ \mu \text{ dominant}}} m(\mu) Symm[\mu],$$

where the $m(\mu) \geq 1$ are integers satisfying $m(\lambda) = 1$. (When G is simply connected then all then $m(\mu) = 1$.)

(b) For $\lambda \in X$ dominant, there are $\epsilon_\mu \in \{\pm 1\}$, for $\mu \leq \lambda$ a dominant character, such that

$$Symm[\lambda] = \sum_{\substack{\mu \leq \lambda \\ \mu \text{ dominant}}} \epsilon_\mu m'(\mu) \chi_\mu|_T,$$

and $\epsilon_\lambda = 1$. Here the $m'(\mu) \geq 1$ are integers satisfying $m'(\lambda) = 1$. (When G is simply connected then all then $m'(\mu) = 1$.)

For a proof, see [10], §3.4.

Recall $X^{\theta +}\subset X$ denotes the subset of θ-invariant dominant characters. If ρ is θ-invariant, let $\overline{\rho}$ denote an extension of ρ to $\overline{G}$. Write

$$\overline{\rho}(g.\theta)=\rho(g)\rho(\theta),\quad g\in G,$$

for some $\rho(\theta)\in GL(V_\rho)$. This is an abuse of notation since the extension is not necessarily unique. $\rho(\theta)$ is only well-defined up to a d^{th} root of unity where $\theta^d=1$. We shall fix an extension in the following definitions. First, we claim that the trace of this endomorphism is, as a function of g, constant on the θ-conjugacy classes. Let

$$\chi_{\overline{\rho}}(g.\theta^i)=\mathrm{trace}(\rho(g)\rho(\theta)^i),\quad g\in G,$$

and, if $\lambda\in X^{\theta +}$ and $\rho=\rho_\lambda$, let

$$\chi^\theta_\lambda(g)=\chi^\theta_\rho(g)=\chi_{\overline{\rho}}(g.\theta),\quad g\in G. \tag{3}$$

This will be called the *θ-character* of ρ. From the fact that $\overline{\rho}(g.\theta)$ is a class function on $G.<\theta>$, it follows that

$$\chi^\theta_\rho(y)=\chi^\theta_\rho(x^{-1}y\theta(x)),\qquad x,y\in G,$$

if ρ is a θ-invariant finite dimensional representation of G.

Lemma 3.7. *(a) For $\lambda\in X^{\theta +}$,*

$$\chi^\theta_\lambda|_{T_\theta}=\sum_{\substack{\mu\leq\lambda\\ \mu\ \theta-\mathrm{dominant}}} m_\theta(\mu)Symm_\theta[\mu]|_{T_\theta},$$

where the $m_\theta(\mu)\geq 1$ are integers satisfying $m_\theta(\lambda)=1$.

(b) For $\lambda\in X^{\theta +}$, there are $\epsilon_\mu\in\{\pm 1\}$, for $\mu\leq\lambda$ a θ-dominant character, such that

$$Symm_\theta[\lambda]|_{T_\theta}=\sum_{\substack{\mu\leq\lambda\\ \mu\ \theta-\mathrm{dominant}}}\epsilon_\mu m'_\theta(\mu)\chi^\theta_\mu|_{T_\theta},$$

and $\epsilon_\lambda=1$. Here the $m'_\theta(\mu)\geq 1$ are integers satisfying $m'_\theta(\lambda)=1$.

Remark 3.8. The "restriction" symbol $|_{T_\theta}$ used above is a slight abuse of notation (since T_θ is not a subset of T), which we hope the reader will pardon. Worst, it is not clear it is well-defined. It must be shown that both the left-hand side of (a) and the right-hand side of (b) above are well-defined. Let ρ denote an irreducible finite dimensional (complex) representation of G and let θ be a quasi-semisimple of G preserving a Borel pair (B,T). If $\rho\cong\rho^\theta$ then, we claim, the restriction of its "twisted character" χ^θ_ρ to T_θ is well-defined. Indeed, if ρ is θ-invariant then the character $\chi_{\overline{\rho}}$ is constant on conjugacy classes. Since $\chi_{\overline{\rho}}(t'.\theta)=\chi_{\overline{\rho}}(tt'\theta(t)^{-1}.\theta)=\chi_{\overline{\rho}}(t't\theta(t)^{-1}.\theta)$, for all $t,t'\in T$.

Proof. The proof of part (a) is analogous to Steinberg's proof in the connected case ([10], §3.4). The restriction of χ_λ^θ to T_θ is a sum of characters (weights) μ in $X^*(T_\theta)$, the multiplicity of each character occurring in the sum is the dimension of the corresponding weight space. If χ' is any weight occurring in this decomposition then there is a $w \in W$ and a dominant $\chi \in X^*(T_\theta)$ (regarded as an element in $X^*(T)^\theta$ by Lemma 3.1) such that $\chi' = w\chi$. Furthermore, these dimensions only depend on the W^θ-equivalence class of μ with respect to $\sim_\theta$ above. By Lemma 3.1, only θ-invariant weights can occur in this sum. The multiplicity with which (the "highest weight") λ occurs in this sum is equal to the multiplicity of λ in $\overline{\rho}_\lambda$. But this is equal to the multiplicity of λ in ρ_λ, by construction (since they have the *same* representation space). Thus $m_\theta(\lambda) = 1$, as desired.

Part (b) follows by inverting the (upper triangular) system of equations given in part (a). ■

Proposition 3.9. *Let θ be as in Lemma 2.7. Assume that G contains no irreducible component of Cartan type A_{2n}. Let G^1 denote the connected component of G^θ and let T^1 denote the connected component of T^θ. The finite set $R|_{T^1}$ forms a root system of the connected semisimple group G^1 with maximal torus T^1.*

Proof. This is a consequence of results in [9], §§12.16-12.19. ■

4. θ-conjugacy classes

Let G and $\overline{G}$ be as in the previous section.

The following theorem, one of our main results, is an extension of a theorem of Steinberg to the non-connected case (see [10], §3.4, Theorem 2).

Theorem 4.1. *Let G be a connected semi-simple group and let θ be as in Lemma 2.7.*

(a) The restriction map

$$k[G.\theta]^G \to k[T.\theta]^{N_G(T.\theta)}$$

is an isomorphism.

(b) The "restriction" map

$$\begin{gathered} k[T.\theta]^{N_G(T.\theta)} \to k[T_\theta]^{W^\theta} \\ f \longmapsto res(f) \end{gathered}$$

is an isomorphism, where $\mathrm{res}(f)(t) = f(t.\theta)$, $t \in T_\theta$ *(see Remark 3.8).*

(c) The functions $\{\chi_\lambda^\theta|_{T_\theta} \mid \lambda \in X^{\theta+}\}$ form a basis for the k-vector space $k[T_\theta]^{W^\theta}$. The functions $\{\chi_\lambda \mid \lambda \in X^{\theta+}\}$ form a basis for the k-vector space $k[G.\theta]^G$.

Now we begin the proof of the theorem.

Proof. (a): Regarding the surjectivity of the restriction map, we need to know that if two elements of $T.\theta$ are G-conjugate then they are $N_G(T.\theta)$-conjugate. This is a consequence of Lemma 2.7.

To prove injectivity, let $f \in k[G.\theta]^G$ be such that $f|_{T.\theta} = 0$. If $x \in G$ is θ-semisimple then there is a $g \in G$ such that $g^{-1}x\theta(g) \in T$ by Lemma 2.7(a). Thus

$$f(x.\theta) = f(g^{-1}x\theta(g).\theta) = 0,$$

since f is a class function. Recall $x \in G$ is θ-semisimple if and only if $x.\theta \in \overline{G}$ is semisimple. Note also the image of the inclusion

$$\overline{G}_{ss} \hookrightarrow \overline{G}$$

is dense (almost all of the elements in $\overline{G}$, regarded as matrices, have distinct eigenvalues). It follows from these facts that f is zero on a dense subset. This implies that the restriction map is injective, which proves (a).

(b): We have $N_G(T.\theta)/T = W^\theta$ by Lemma 2.6. Therefore,

$$k[T.\theta]^{N_G(T.\theta)} = k[T.\theta]^{W^\theta T} \cong k[T_\theta]^{W^\theta}.$$

This proves (b).

(c): The first statement is a corollary of parts (a), (b), and the proof of Theorem 2(a) in §3.4 of [10]. The second statement follows from the first and parts (a), (b).

This completes the proof of the theorem. ∎

5. Some corollaries

We list some corollaries of Theorem 4.1 above. These are all analogs of results of Steinberg in the non-connected case.

Let G and $\overline{G}$ be as in the previous section.

Definition 5.1. We call G *θ-simply connected* (resp., *θ-adjoint*) and call $\overline{G}$ *simply connected* (resp., *adjoint*) if $X^\theta = P^\theta$ (resp., $X^\theta = Q^\theta$).

The following result can be proven by modifying the proof of Theorem 2, §3.4, in [10] and using the above proof.

Corollary 5.2. *Assume G is θ-simply connected and let $\lambda_1, ..., \lambda_n$ denote a set of θ-fundamental weights. Then $\{\chi_{\lambda_i}|_{T_\theta} \mid 1 \leq i \leq n\}$ freely generates $k[T_\theta]^{W^\theta}$ as a k-algebra.*

Define $k[G_{\theta-ss}.\theta]$ to be the vector space of functions on $G.\theta$ restricted to $G_{\theta-ss}.\theta$. The result below is a component of the proof of Theorem 4.1(a).

Corollary 5.3. *The map*

$$\begin{gathered} k[G.\theta]^G \to k[G_{\theta-ss}.\theta]^G \\ f \longmapsto res(f) \end{gathered}$$

is an isomorphism of vector spaces, where $res(f)$ denotes the restriction map.

Corollary 5.4. *The θ-semisimple conjugacy classes of G are in 1-1 correspondence with the elements of T_θ/W^θ.*

Remark 5.5. This Corollary is the twisted analog of Corollary 2, §3.4 in [10]. See also Remark 2.8 above.

Corollary 5.6. *Let $x = x_0.\theta, y = y_0.\theta \in G.\theta$ be semisimple elements. The following are equivalent.*

(a) x, y are G-conjugate,

(b) $\chi_\rho(\overline{x}_0) = \chi_\rho(\overline{y}_0)$, for all $\rho = \rho_\lambda$, $\lambda \in X^{\theta+}$. Here $\overline{x}_0, \overline{y}_0 \in T_\theta$ denote the image of $\mathcal{C}_\theta(x_0) \cap T, \mathcal{C}_\theta(y_0) \cap T$ in T_θ (which exists by Lemma 2.7).

Proof. [3] Follows immediately from the second statement in Theorem 4.1(c). ■

Corollary 5.7. *If $f \in k[G.\theta]^G$ and $x \in G$ then $f(x.\theta) = f(x_{\theta-ss}.\theta)$, for $x \in G$.*

Proof. This follows from the second statement in Theorem 4.1(c) and the fact that $\chi_\lambda(x.\theta) = \chi_\lambda(x_{\theta-ss}.\theta)$, for all $x \in G$ and $\lambda \in X^{\theta+}$. ■

Corollary 5.8. *A G-conjugacy class in $G.\theta$ is closed if it is semisimple.*

Proof. We pick a basis of the vector space V, in the notation of §2 where $\overline{G} \subset GL(V)$. We may identify each $g \in \overline{G}$ with a matrix in $GL(V)$.

For the proof, use the obvious twisted analog of the proof of Corollary 5 in §3.4, p. 92 of [10] (which relies on Corollary 5.6 above). In other words, fix a (semisimple) element $x_0.\theta \in G.\theta$. Let m_0 denote the minimal polynomial of $x_0.\theta$ and let

$$S = \{x.\theta \in G.\theta \mid \chi^\theta_\lambda(x_0) = \chi^\theta_\lambda(x), \text{ for all } \lambda \in X^{+^\theta}, \text{ and } m_0(x.\theta) = 0\}.$$

S is Zariski closed and contains the conjugacy class of $x_0.\theta$. Now let $x.\theta \in S$. It is semisimple since its minimal polynomial has distinct roots (it divides m_0, which has distinct roots by definition). The hypothesis to Corollary 5.6 therefore holds and implies the statement of the corollary. ■

6. Character relations and endoscopic groups

Let θ be as in Lemma 2.7. As in Lemma 3.1, we fix an isomorphism

$$\psi : T^\theta \to T_\theta.$$

The goal of this section is to show that, if G is a simple, simply connected (hence θ-simply connected) group and if θ is an automorphism of order d preserving a splitting $(B, T, \{X\})$ then there is a simple connected group H associated to G, θ such that

[3] I thank the referee for this short proof.

1. $X^{\theta+}$ is in natural 1-1 correspondence (defined in Lemma 3.1) with the dominant characters of H, X_H^+, which we denote by

$$\lambda \leftrightarrow \lambda_H,$$

2. there are distinct $\lambda_{H,i} \in X_H^+$ such that we have a character relation

$$\chi_\lambda^\theta(\mathcal{A}_\theta(\eta(t).\theta)) = \sum_{i=1}^{N} \zeta_i \chi_{\lambda_{H,i}}(t), \tag{4}$$

for all $t \in T_H$, where χ_λ^θ is as in (3) above, T_H is a regular torus, and $\mathcal{A}_\theta$ is a map (to be defined below) from the semisimple conjugacy classes in $G.\theta$ to the semisimple conjugacy classes of H. Here the $\zeta_i \in \mathbb{C}$ (possibly all zero) depend on the extension of ρ_λ from G to $G \rtimes <\theta>$ chosen in the definition of the θ-character.

As in [4], we have the following definition.

Definition 6.1. A group H as above is called a *θ-endoscopic group.*

Define H to be the group $H = G^\theta = \{g \in G \mid \theta(g) = g\}$. Since G is simply connected, H is connected, by Steinberg's Theorem 8.1 in [11]. H has maximal torus $T_H = T^\theta$, root lattice Q_H, character lattice $X_H = X^*(T_H)$, and weight lattice P_H, such that

$$Q_H = \mathbb{Z}[R(B,T)|_{T^\theta}] \subset X_H \subset P_H = Q_H^\perp,$$

by Proposition 3.9. Furthermore, the Dynkin diagram of H is the "folded" Dynkin diagram of G (see [9], §12.18 and Theorem 12.19). We have $T_H \cong T_\theta$, $W_H \cong W^\theta$ (see [11], §8). This implies H has property (1) of an endoscopic group. Furthermore, there is a 1-1 correspondence

$$\mathcal{A}_\theta : \mathcal{C}_G(G_{\theta-ss}.\theta) \to \mathcal{C}_H(H_{ss}), \tag{5}$$

defined using the correspondences

$$\mathcal{C}_H(H_{ss}) \leftrightarrow T_H/W_H,$$

$$\mathcal{C}_G(G_{\theta-ss}.\theta) \leftrightarrow T_\theta/W^\theta,$$

and the non-canonical isomorphism

$$T_H/W_H \cong T_\theta/W^\theta. \tag{6}$$

The following theorem is our other main result.

Theorem 6.2. *Assume G and θ are as in Theorem 4.1 above. In addition, assume G is simply connected. Let $H = G^\theta$ be as defined above. There is an isomorphism of vector spaces*

$$k[G.\theta]^G \to k[H]^H,$$

induced by the restriction maps $k[G.\theta]^G \to k[T_\theta]^{W^\theta}$, $k[H]^H \to k[T_H]^{W_H}$, and (6).

Proof. This follows from Theorem 4.1 and (6):

$$k[G.\theta]^G \to k[T.\theta]^{N_G(T.\theta)} \to k[T_\theta]^{W^\theta}.$$

∎

Remark 6.3. The equality claimed in (4) above, which is property (2) of an endoscopic group, is a special case of Lemma 3.7, using the definition of X_H^+.

7. Irreducible representations of $\overline{G}$, G simple

In this section, we use Dynkin diagrams to describe those irreducible finite dimensional representations of G which extend to representations of $\overline{G}$. This forms an important component of the complete description, which may be found in Remark 3.3 above. No proofs are given in this section. All statements are either proven in [2] or may be derived by modifying arguments there.

We remark that a relationship between the automorphisms of G and the automorphisms of the Dynkin diagram of G is given by Proposition 1.4.1 in [7].

G simply connected of type A_n

I thank the referee for pointing out that if n is even the restrictions of the roots in this case do not form a restricted root system in the sense of Proposition 3.9.

Here

$$\lambda_1, \quad \text{the } 1-\text{st fundamental weight} \leftrightarrow \text{the irreducible repn } k^{n+1},$$

$$\lambda_2, \quad \text{the } 2-\text{nd fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^2 k^{n+1},$$

and so on. The last node on the far right end is associated to the contragrediant of the standard representation:

$$\lambda_n, \quad \text{the n}-\text{th fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^n k^{n+1}.$$

Let θ denote the automorphism $\theta(g) = J\ {}^tg^{-1}J$, where J denotes the skew-diagonal matrix whose skew-diagonal (top right to bottom left) is given by $1, -1, ...,$ $(-1)^{n+1}$. Then θ preserves the usual Borel pair (B,T) where B denotes the upper triangular subgroup and T the diagonal torus. Write the simple roots

$\Delta = \{\alpha_1, ..., \alpha_n\}$ of (B,T) as usual, $\alpha_1(t) = t_1/t_2, ..., \alpha_n(t) = t_n/t_{n+1}$, where $t = diag(t_1, ..., t_{n+1})$. If we write the Dynkin diagram as usual, with α_i in place of λ_i above, then θ acts on the roots by

$$\theta : \alpha_i \leftrightarrow \alpha_{n+1-i},$$

and on the weights by

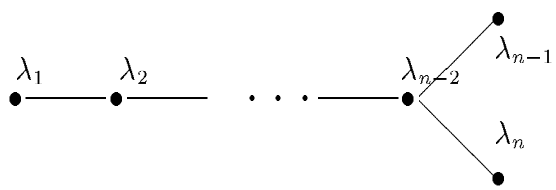
$$\theta : \lambda_i \leftrightarrow \lambda_{n+1-i},$$

for $1 \leq i \leq n$. The θ-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_i + \theta(\lambda_i) = \lambda_i + \lambda_{n+1-i}, & i < \frac{n+1}{2}, \\ \lambda_i, \quad i = \frac{n+1}{2}, \end{cases}$$

where $1 \leq i \leq \frac{n+1}{2}$.

G simply connected of type D_n



$$\lambda_1, \quad \text{the } 1-\text{st fundamental weight} \leftrightarrow \text{the irreducible repn } k^{2n},$$

$$\lambda_2, \quad \text{the } 2-\text{nd fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^2 k^{2n},$$

and so on. The third to the last node on the far right end is associated to the representation:

$$\lambda_{n-2}, \quad \text{the n} - 1 - \text{st fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^{n-1} k^{2n}.$$

The top node on the far right is associated to the Spin$^+$ representation:

$$\lambda_{n-1}, \quad \text{the n} - \text{th fundamental weight} \leftrightarrow \text{the irreducible repn Spin}^+.$$

The bottom node on the far right is associated to the Spin$^-$ representation:

$$\lambda_n, \quad \text{the n} - \text{th fundamental weight} \leftrightarrow \text{the irreducible repn Spin}^-.$$

Assume $n > 3$. If $n > 4$ then the only non-trivial diagram automorphism is that which exchanges the two nodes on the far right end and leaves the others fixed. Let θ denote this automorphism (even when $n = 4$). In this case, the θ-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_i + \theta(\lambda_i) = \lambda_i + \lambda_{n+1-i}, & i = n-1, \\ \lambda_i, \quad i < n-1, \end{cases}$$

where $1 \leq i \leq n-1$.

Now assume $n = 4$ and let θ denote the diagram automorphism such that

$$\theta : \alpha_1 \longmapsto \alpha_4 \longmapsto \alpha_3, \quad \theta(\alpha_2) = \alpha_2.$$

Using the fact that

$$\begin{aligned} \lambda_1 &= \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{2}\alpha_4, \\ \lambda_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\ \lambda_3 &= \tfrac{1}{2}\alpha_1 + \alpha_2 + \alpha_3 + \tfrac{1}{2}\alpha_4, \\ \lambda_4 &= \tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 + \alpha_4, \end{aligned}$$

we find that

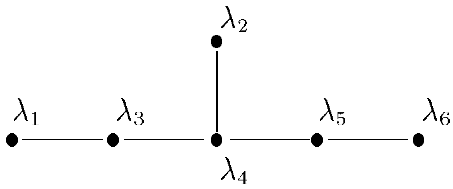
$$\theta : \lambda_1 \longmapsto \lambda_4 \longmapsto \lambda_3, \quad \theta(\lambda_2) = \lambda_2.$$

Therefore, the θ-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_1 + \lambda_4 + \lambda_3, & i = 1 \\ \lambda_2, & i = 2, \end{cases}$$

where $1 \leq i \leq 2$.

G simply connected of type E_6



Let θ denote an automorphism of G which acts on the roots by

$$\theta : \alpha_1 \leftrightarrow \alpha_6, \quad \alpha_3 \leftrightarrow \alpha_5, \quad \alpha_4 \leftrightarrow \alpha_4, \quad \alpha_2 \leftrightarrow \alpha_2.$$

Then from the table 1, [3], p. 69, we find that

$$\theta : \lambda_1 \leftrightarrow \lambda_6, \quad \lambda_3 \leftrightarrow \lambda_5, \quad \lambda_4 \leftrightarrow \lambda_4, \quad \lambda_2 \leftrightarrow \lambda_2.$$

The θ-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_1 + \lambda_6, & i = 1, \\ \lambda_3 + \lambda_5, & i = 3, \\ \lambda_2, & i = 2, \\ \lambda_4, & i = 4, \end{cases}$$

where $1 \leq i \leq 4$.

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