

Geodesic loops

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Abstract. Using new tangential objects, which we call Λ -algebras, we obtain for geodesic loops (L, ∇, e) statements that are analogous to Lie's theorems. We prove that any geodesic loop L defines in the tangential space $T_e L$ a unique Λ -algebra, and that to any finite dimensional real Λ -algebra F there exists a geodesic loop L , the Λ -algebra of which is isomorphic to F . Geodesic loops which have isomorphic Λ -algebras, are locally isomorphic.

If the local loop L is diassociative then to the Λ -algebra of L there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to L .

Introduction

A.I. Malcev perceived that the fundamental ideas of Sophus Lie to assign to any local Lie group G a tangential object -its Lie algebra- which determines G in a unique way can be extended to non-associative structures. In particular with any differentiable Moufang loop L is associated a Malcev algebra which determines L uniquely up to coverings (cf. [9] Theorem 4.4). Moreover, in the mean time the theory of Lie Moufang loops reached the same level as the theory of Lie groups (cf. [3], [5], [8]). The most general class of local analytical loops for which we know that they are determined by their tangential objects -by their Bol algebras- in a unique way are the Bol loops (cf. [7] pp. 424-425). In general, the differentiable loops have as tangential objects the Akinis algebras, but an Akinis algebra corresponds to a variety of local differentiable loops the algebraic properties of which show great differences (cf. [2] pp. 240-241).

The aim of this paper is to show that there exist suitable tangential objects for very wide classes of local analytical loops L which determines L in a unique way; concretely we demonstrate this for the class of geodesic loops which respect to linear connections the curvature of which is zero; these tangential objects have in general infinitely many binary operations. Since a differentiable local loop is strongly left alternative if and only if it can be represented as the opposite loop of a geodesic loop with respect to a linear connection with curvature zero, the local

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analytic Bol loops are a proper subclass of geodesic loops considered in our paper (cf. [11] Proposition 4.7).

The tangential object $T_e L$ of a geodesic loop L with respect to a linear connection with vanishing curvature is called a Λ -algebra. The infinitely many binary operations $\Lambda_{(p,1)}(x, y) : T_e L \times T_e L \rightarrow T_e L$ are linear in y and p -homogeneous in x ; moreover, they satisfy certain symmetry conditions. In the case that the local loop L is diassociative to the Λ -algebra of L there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to L .

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1. Geodesic loops and their Λ -algebras

Definition 1.1. Let L be an analytic manifold and B an open neighbourhood of the identity e of L . Let $*, /, \backslash : B \times B \rightarrow L$ be analytic mappings. The manifold L is called a local analytic loop if the analytic mappings $*, /, \backslash : B \times B \rightarrow L$ satisfy the following identities:

1. $x * e = e * x = x/e = e \backslash x = x$ for all $x \in B$.
2. $(x/y) * y = x, \quad y * (y \backslash x) = x,$
 $(x * y)/y = x, \quad y \backslash (y * x) = x$ for all $x, y \in B$.

provided the left sides of these equations are defined.

Definition 1.2. Let L be a differentiable manifold equipped with an affine connection ∇ and let e be a point in L . A local analytic loop (L, ∇, e) is called a geodesic loop if in a normal neighbourhood U of $e \in L$ the multiplication of L is given by

$$x * y = \exp_y \circ \tau_{e,y} \circ \exp_e^{-1}(x)$$

for all $x, y \in U$, where $\tau_{e,y}$ denotes the parallel translation $T_e L \rightarrow T_y L$ along the unique geodesic segment, contained in U , between e and y (cf. [4], p. 160 and [7], p. 369).

Definition 1.3. Let V be an n -dimensional real or complex vector space. V is called a Λ -algebra if there exists for every natural number p an algebraic map $\Lambda_{(p,1)} : V \times V \rightarrow V$ such that the following properties are satisfied:

1. $\Lambda_{(p,1)}(x, \lambda y) = \lambda \Lambda_{(p,1)}(x, y)$ for all $\lambda \in K$ and $x, y \in V$.
2. $\Lambda_{(p,1)}(\lambda x, y) = \lambda^p \Lambda_{(p,1)}(x, y)$ for all $\lambda \in K$ and $x, y \in V$.
3. $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x, y)$ converges on a neighbourhood N of 0 in V .

Remark 1.4. The coordinate functions $\Lambda_{(p,1)}^i(x, y), i = 1, \dots, n$, of $\Lambda_{(p,1)}(x, y)$ are homogeneous polynomials of degree $p + 1$ which are linear in the coordinates of y and homogeneous polynomials of degree p in the coordinates of x .

The Λ -algebras are suitable tangential objects for geodesic loops (L, ∇, e) for which the curvature tensor is 0, in order to obtain analogous statements to Lie's theorems.

Theorem 1.5. (Lie’s First Theorem) *Any geodesic loop L defines in the tangential space $T_e L$ a unique Λ -algebra.*

Proof. The 1-parameter subgroups of a geodesic loop (L, ∇, e) are geodesic lines through the identity e of L with respect to the linear connection ∇ . Hence the canonical coordinate system for L (which is determined by the 1-parameter subgroups) is a normal coordinate system of the connection ∇ in a normal neighbourhood $U \subset \mathbb{R}^n$. Since the curvature of ∇ is zero every parallel translation with respect to ∇ can be given in U by the independent parallel vector fields $A_1(x), \dots, A_n(x)$, ($i = 1, \dots, n$), where $A_i(0) = e_i$ yield the canonical basis of \mathbb{R}^n . The parallel translation $\tau_{e,y}$ is determined by $e_i \mapsto A_i(y)$ ($i = 1, \dots, n$). Hence the map $\sum_{i=1}^n \xi^i e_i \mapsto \sum_{i=1}^n A_i(y) \xi^i$ gives the tangent map $T_e \rho_y$ for the right translation $\rho_y : x \rightarrow x * y$. The differential equations of the coordinate functions $y^i(t)$ for the geodesic line $y(t)$ with the initial values $y(0) = y_0$ and $y'(0) = \sum_{j=1}^n A_j(y_0) \xi^j$ have the form

$$y'^i(t) = \sum_{j=1}^n A_j^i(y(t)) \xi^j, \tag{1}$$

where A_j^i denote the coordinate functions of the vector fields A_j and ξ^j are the coordinates with respect to the basis $A_j(y_0)$. Since in the coordinate system of U the geodesic lines are the euclidean lines $(\xi^1, \dots, \xi^n)t$, $t \in \mathbb{R}$, equation (1) implies that the coordinate functions $A_j^i(x)$ of A_j satisfy the identities

$$\xi^i = \sum_{j=1}^n A_j^i((\xi^1, \dots, \xi^n)t) \xi^j.$$

The power series of the coordinate functions $A_j^i(x)$ through the point $(0, \dots, 0)$ can be written as

$$\xi^i = \sum_{j=1}^n [A_j^i(0) + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p=1}^n \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0) \xi^{j_1} \dots \xi^{j_p}] \xi^j.$$

Since $A_j^i(0) = \delta_j^i$ we obtain

$$\xi^i = \xi^i + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p=1}^n \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0) \xi^{j_1} \dots \xi^{j_p} \xi^{j_{p+1}}.$$

Consequently the coefficients of this expansion satisfy

$$\sum_{\sigma \in \mathbb{Z}_{p+1}} \frac{\partial^p A_{j^{(p+1)\sigma}}^i}{\partial x^{j_1 \sigma} \dots \partial x^{j_p \sigma}}(0) = 0, \tag{2}$$

where \mathbb{Z}_{p+1} is the cyclic permutation group of order $p + 1$.

In the tangential space of the identity $e \in L$ we define now the binary operations $\Lambda_{(p,1)}$ as follows:

$$\Lambda_{(1,1)}(x, y) = \left(\sum_{j=1, h_1=1}^n \frac{\partial A_j^i}{\partial x^{h_1}}(0) x^{h_1} y^j \right)_{i=1}^n$$

$$\Lambda_{(2,1)}(x, y) = \left(\sum_{j=1, h_1=1, h_2=1}^n \frac{1}{2!} \frac{\partial^2 A_j^i}{\partial x^{h_1} \partial x^{h_2}}(0) x^{h_1} x^{h_2} y^j \right)_{i=1}^n$$

$$\Lambda_{(p,1)}(x, y) = \left(\sum_{j=1, h_1=1, \dots, h_p=1}^n \frac{1}{p!} \frac{\partial^p A_j^i}{\partial x^{h_1} \dots \partial x^{h_p}}(0) x^{h_1} \dots x^{h_p} y^j \right)_{i=1}^n.$$

The i -th coordinate of the vector $\Lambda_{(p,1)}(x, y)$ is

$$\Lambda_{(p,1)}^i(x, y) = \sum_{j=1, h_1=1, \dots, h_p=1}^n \frac{1}{p!} \frac{\partial^p A_j^i}{\partial x^{h_1} \dots \partial x^{h_p}}(0) x^{h_1} \dots x^{h_p} y^j.$$

This function is linear in the coordinates of y since it is depending only of the sum of the coordinates of y . Hence $\Lambda_{(p,1)}(x, y)$ is linear in y . We have

$$\Lambda_{(p,1)}^i(\lambda x, y) = \sum_{j=1, h_1=1, \dots, h_p=1}^n \frac{1}{p!} \frac{\partial^p A_j^i}{\partial x^{h_1} \dots \partial x^{h_p}}(0) \lambda x^{h_1} \dots \lambda x^{h_p} y^j = \lambda^p \Lambda_{(p,1)}^i(x, y).$$

Hence the second condition for a Λ -algebra is satisfied. The coordinate functions of the product $(x^1, \dots, x^n) * (y^1, \dots, y^n)$ are given by $A_j^i(y)$ and its partial derivaties (cf.[10] pp.309-312). Since L is analytic loop and the series $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x, y)$ is a subseries of the power expansion of the product $x * y$ the condition 3. in the definition of a Λ -algebra is valid. ■

Theorem 1.6. (Lie's Third Theorem) *To any finite dimensional real Λ -algebra F there exists a geodesic loop L , the Λ -algebra of which is isomorphic to F . The geodesic loops which have isomorphic Λ -algebras, are locally isomorphic.*

Proof. Let F be an n -dimensional real Λ -algebra. The convergent series $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x, y)$ determines a function $f = (f^1, \dots, f^n)$ from $F \times F$ into F . The function f is linear in y since the operations $\Lambda_{(p,1)}(x, y)$ are linear in y . Hence the i -th coordinate function $f^i(x, y)$ has the form

$$f^i(x^1, \dots, x^n, y^1, \dots, y^n) = \sum_{j=1}^n f_j^i(x) y^j$$

with suitable real analytic functions f_j^i . If we develop the functions f_j^i into Taylor series in the point zero we obtain

$$f_j^i(x) = \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{h_1=1, \dots, h_p=1}^n \frac{\partial^p f_j^i}{\partial x^{h_1} \dots \partial x^{h_p}}(0) x^{h_1} \dots x^{h_p}.$$

Since the real analytic function f is described uniquely by a convergent series we can write

$$\Lambda_{(p,1)}(x, y) = \left(\sum_{j=1, h_1=1, \dots, h_p=1}^n \frac{1}{p!} \frac{\partial^p A_j^i}{\partial x^{h_1} \dots \partial x^{h_p}}(0) x^{h_1} \dots x^{h_p} y^j \right)_{i=1}^n,$$

where $A_j^i = f_j^i$. The coordinate functions A_j^i determine the vector fields A_j which are defined as $A_j : y \rightarrow (A_j^1(y), \dots, A_j^n(y))$. Considering F as the affine space \mathbb{R}^n we can assign to any point y of F a tangent space $T_y F = \sum_{i=1}^n \eta^i A_i(y), \eta_i \in \mathbb{R}$ and form the tangent bundle TF . Any vector $v \in T_0 F, v = \sum_{i=1}^n \eta^i e_i$, determines a vector field Y_v putting $Y_v(y) = \sum_{i=1}^n \eta^i A_i(y)$.

We define now a covariant derivation ∇ , such that the vector fields A_i become parallel vector fields with respect to ∇ . Hence in the expression

$$\nabla_X Y = \nabla_X \left(\sum_{i=1}^n Y^i A_i \right) = \sum_{i=1}^n (X(Y^i)) A_i + \sum_{i=1}^n Y^i \nabla_X A_i$$

we put $\sum_{i=1}^n Y^i \nabla_X A_i = 0$ for any vector field X . For the curvature tensor $R(X, Z)Y$ with respect to ∇ we have

$$\begin{aligned} R(X, Z)Y &= \nabla_X \nabla_Z \left(\sum_{i=1}^n Y^i A_i \right) - \nabla_Z \nabla_X \left(\sum_{i=1}^n Y^i A_i \right) - \nabla_{[X, Z]} \left(\sum_{i=1}^n Y^i A_i \right) = \\ &= \nabla_X \left(\sum_{i=1}^n (Z(Y^i)) A_i \right) - \nabla_Z \left(\sum_{i=1}^n (X(Y^i)) A_i \right) - \sum_{i=1}^n ([X, Z](Y^i)) A_i = \\ &= X \left(\sum_{i=1}^n (Z(Y^i)) A_i \right) - Z \left(\sum_{i=1}^n (X(Y^i)) A_i \right) - \sum_{i=1}^n ([X, Z](Y^i)) A_i. \end{aligned}$$

Since $[X, Z] = XZ - ZX$ we can write $[X, Z](Y^i) = X(Z(Y^i) - Z(X(Y^i)))$. It follows

$$R(X, Z)Y = X \left(\sum_{i=1}^n (Z(Y^i)) A_i \right) - Z \left(\sum_{i=1}^n (X(Y^i)) A_i \right) - \sum_{i=1}^n ([X, Z](Y^i)) A_i = 0.$$

The parallel translations $\tau_{0,y}$ are determined by $\tau_{0,y}(e_i) = A_i(y)$. Since the curvature tensor R with respect to ∇ is zero the parallel translations do not depend on the path. Let exp be the exponential map belonging to the linear connection ∇ . We have $exp_y : T_y F \rightarrow F : exp_y(v_y) = \gamma(1)$ where γ is the geodesic curve on F with respect to ∇ satisfying the conditions $\gamma(0) = y$ and $\gamma'(0) = v_y$. The multiplication $x * y = exp_y \circ \tau_{e,y} \circ exp_e^{-1}(x)$ for $x, y \in F$ defines on F a geodesic loop L with 0 as the identity. The Λ -algebra of L is uniquely determined by the parallel translations $\tau_{0,y}$ and hence by the vector fields $A_i, i = 1, \dots, n$, as well as by the derivatives of their coordinate functions. ■

A real algebra $(C, [., .])$ is called a binary Lie algebra if any two elements of C generate a Lie subalgebra.

To the Λ -algebras of Lie algebras and of binary Lie algebras there correspond subseries of the Campbell-Hausdorff-series. These subseries are determined by the Lie algebras or binary Lie algebras.

Theorem 1.7. *Let C a binary Lie algebra and $(b_{2n})_{n \in \mathbb{N}}$ the sequence of the even Bernoulli numbers. The operations $\Lambda_{(p,1)} : C^2 \rightarrow C$ given by*

$$\begin{aligned}\Lambda_{(1,1)}(X, Y) &= \frac{1}{2}[X, Y] \\ \Lambda_{(2,1)}(X, Y) &= \frac{1}{12}[X, [X, Y]] \\ \Lambda_{(2n,1)}(X, Y) &= \frac{1}{(2n)!}b_{2n}[X, \dots, [X, Y], \dots] \\ \Lambda_{(2n+1,1)}(X, Y) &= 0\end{aligned}$$

define on the vector space C a Λ -algebra $\Lambda(C)$.

Proof. The operations $\Lambda_{(p,1)}$ are linear in Y . Moreover, we have

$$\Lambda_{(p,1)}(\lambda X, Y) = \lambda^p \Lambda_{(p,1)}(X, Y)$$

for any $\lambda \in \mathbb{R}$. The series $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(X, Y)$ converges since it is a subseries of the Hausdorff-Campbell series (cf. [1] pp. 201-202). ■

An analytic loop L is called diassociative if every two elements generate a subgroup of L .

Theorem 1.8. *The Λ -algebra F of a local diassociative analytic loop $L = \exp(F)$ coincides with the Λ -algebra $\Lambda(C)$, where C is the binary Lie algebra belonging to L .*

Proof. According to Malcev, theorems 1 and 2 (cf. [6] pp. 570-571), the tangential algebra of L is a binary Lie algebra C . The power series of the multiplication of L is the classical Hausdorff-Campbell series. Let $\Lambda_{(p,1)}, p \in \mathbb{N}$, be the operations of the Λ -algebra of L and $\Lambda_{(p,1)}^C, p \in \mathbb{N}$, be the operations for the Λ -algebra $\Lambda(C)$. Since $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x, y)$ is a subseries of the loop multiplication and

$$\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x, y) = \sum_{p=1}^{\infty} \Lambda_{(p,1)}^C(x, y)$$

we have $\Lambda_{(p,1)} = \Lambda_{(p,1)}^C$. ■

Theorem 1.9. *The local analytic loop L the Λ -algebra of which is $\Lambda(C)$ for a binary Lie algebra C is diassociative and C is the tangential algebra of L in sense of Malcev.*

Proof. Since $\Lambda(C)$ is the Λ -algebra of L , the series

$$\sum_{p=1}^{\infty} \Lambda_{(p,1)}^C(x, y) \tag{3}$$

is a subseries of the series of the multiplication $(x, y) \mapsto x * y : L \times L \rightarrow L$ such that $x * y = x + y + \sum_{p,q=1}^{\infty} \Lambda_{(p,q)}^C(x, y)$ and the operations $\Lambda_{(p,q)}^C$ are uniquely determined by $\Lambda_{(p,1)}^C$ through a recursive formula (G) in (cf. [10] p. 311). Since (3) is a subseries of the classical Hausdorff-Campbell series too, hence the multiplication $(x, y) \mapsto x * y$ is described by the classical Hausdorff-Campbell formula and there exists a diassociative loop having the same subseries (3) as L . The multiplication of this loop coincides with the multiplication of L and C is the binary Lie algebra in sense of Malcev (cf. [6] pp. 570-571). ■

Remark 1.10. Theorems 4 and 5 remain valid also for monoalternative analytic loops since any such loop is diassociative (cf. [12]).

References

- [1] Bourbaki, N., "Lie groups and Lie algebras, Chapter 1-3," Springer Verlag, 1989.
- [2] Hofmann, K. H., and K. Strambach, *Topological and analytical loops*, in "Quasigroups and Loops: Theory and Applications" (Chein, O., Pflugfelder, H.O., and Smith, J.D.H.), Sigma Series in Pure Math. 8, Heldermann Verlag, 1990, 205–262.
- [3] Kerdman, F. S., *Moufang loops in the large (Russian)*, Algebra i Logika **18** (1979), 523–555.
- [4] Kikkawa, M., *Geometry of homogeneous Lie loops*, Hiroshima Mathematical Journal **5** (1975), 141–179.
- [5] Kuzmin, E. N., *The connection between Malcev algebras and analytic Moufang loops (Russian)*, Algebra i Logika **10** (1971), 3–22.
- [6] Malcev, A. I., *Analytical loops (Russian)*, Matematicheskii Sbornik **36** (1955), 569–576.
- [7] Miheev, P. O., and L. V. Sabinin, *Quasigroups and Differential Geometry*, Chapter XII in "Quasigroups and Loops: Theory and Applications" (Chein, O., Pflugfelder, H.O., and Smith, J.D.H.), Sigma Series in Pure Math. 8, Heldermann Verlag, 1990, pp. 357–430.
- [8] Nagy, P. T., *Extension of local loop isomorphisms*, Monatshefte Math. **112** (1991), 221–225.
- [9] —, *Moufang loops and Malcev algebras*, Journal of Lie Theory (=Seminar Sophus Lie) **3** (1992), 65–68.
- [10] Nagy, P. T., and K. Strambach, *Loops, their cores and symmetric spaces*, Israel Journal of Mathematics **105** (1998), 285–322.
- [11] —, "Sections in groups: A Lie Theory of smooth loops," Preprint pp. 267.
- [12] Sabinin, L. V., *On the diassociativity of smooth monoalternative loops*, Russian Mathematical Surveys **51** (1996), 747–749.