Infinite Sets of $b$-Additive and $b$-Multiplicative Ramanujan-Hardy Numbers

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Abstract

Let $b$ a numeration base. A $b$-additive Ramanujan-Hardy number $N$ is an integer for which there exists at least one integer $M$, called the additive multiplier, such that the product of $M$ and the sum of base-$b$ digits of $N$, added to the reversal of the product, gives $N$. We show that for any $b$ there exist infinitely many $b$-additive Ramanujan-Hardy numbers and infinitely many additive multipliers. A $b$-multiplicative Ramanujan-Hardy number $N$ is an integer for which there exists at least an integer $M$, called the multiplicative multiplier, such that the product of $M$ and the sum of base-$b$ digits of $N$, multiplied by the reversal of the product, gives $N$. We show that for $b \equiv 4 \pmod{6}$, and for $b = 2$, there exist infinitely many $b$-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers. If $b$ even, $b \equiv 0 \pmod{3}$ or $b \equiv 2 \pmod{3}$, we show there exist infinitely many numeration bases for which there exist infinitely many $b$-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers.

These results completely answer two questions and partially answer two other questions asked in a previous paper of the author.

1 Introduction

Let $b \geq 2$ be a numeration base. In Nițică [6], motivated by some properties of the taxicab number, 1729, we introduce the classes of $b$-additive Ramanujan-Hardy (or $b$-ARH) numbers
and \( b \)-multiplicative Ramanujan-Hardy (or \( b \)-MRH) numbers. The first class consists of numbers \( N \) for which there exists at least an integer \( M \), called the additive multiplier, such that the product of \( M \) and the sum of base-\( b \) digits of \( N \), added to the reversal of the product, gives \( N \). The second class consists of numbers \( N \) for which there exists at least an integer \( M \), called the multiplicative multiplier, such that the product of \( M \) and the sum of base-\( b \) digits of \( N \), multiplied by the reversal of the product, gives \( N \).

It is asked [6, Question 6] if the set of \( b \)-ARH numbers is infinite and it is asked [6, Question 8] if the set of additive multipliers is infinite. It is shown [6, Theorems 12 and 15] that the answer is positive if \( b \) is even. The case \( b \) odd is left open. It is asked [6, Question 7] if the set of \( b \)-MRH numbers is infinite and it is asked [6, Question 9] if the set of multiplicative multipliers is infinite. It is shown [6, Theorem 30] that the answer is positive if \( b \) is odd. The case \( b \) even is left open.

We recall that \textit{Niven (or Harshad) numbers} are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied. See, for instance, Cai [1], Cooper and Kennedy [2], De Koninck and Doyon [3], and Grundman [4]. Of interest are also \( b \)-Niven numbers, which are numbers divisible by the sum of their base-\( b \) digits. See, for example, Fredricksen, Ionașcu, Luca, and Stănică [5]. A \( b \)-MRH-number is a \( b \)-Niven number. High degree \( b \)-Niven numbers are introduced in [7].

The goal of this paper is to show that, for any numeration base, there exist infinitely many \( b \)-ARH numbers and infinitely many distinct additive multipliers. We also show that, for \( b \equiv 4 \pmod{6} \) and for \( b = 2 \), there exist infinitely many \( b \)-MRH numbers, and infinitely many distinct multiplicative multipliers. If \( b \) even, \( b \equiv 0 \pmod{3} \) or \( b \equiv 2 \pmod{3} \), we show there are infinitely many numeration bases for which there exist infinitely many \( b \)-multiplicative Ramanujan-Hardy numbers and infinitely many multiplicative multipliers. These results completely answer the first two questions from [6] revisited above, and partially answer the other two. We observe that a trivial example of infinitely many \( b \)-MRH numbers is given by the powers of \( 10 \). Our examples have at least two digits different from zero. Finding infinitely many \( b \)-MRH numbers with all digits different from zero remains an open question.

Our results about \( b \)-ARH numbers also give solutions to the Diophantine equation \( N \cdot M = \text{reversal}(N \cdot M) \). Motivated by this link, we show that the Diophantine equation has solution for all integers \( N \) not divisible by the numeration base \( b \). We do not know how to answer the following related question:

**Question 1.** Does there exist, for any integer \( N \), an integer \( M \) such that \( N \cdot M \) is a \( b \)-ARH number (or a \( b \)-MRH number, or a \( b \)-Niven number)?

Our final result shows that for any string of digits \( I \) there exist infinitely many \( b \)-Niven numbers that contain \( I \) in their base-\( b \) representation. We do not know a similar result for the classes of \( b \)-ARH and \( b \)-MRH numbers.
2 Statements of the main results

Let \( s_b(N) \) denote the sum of base-\( b \) digits of integer \( N \). If \( x \) is a string of digits, let \( x^\wedge k \) denote the base-10 integer obtained by repeating \( x \) \( k \)-times. Let \( [x]_b \) denote the value of the string \( x \) in base \( b \). If \( N \) is an integer, let \( N^R \) denote the reversal of \( N \), that is, the number obtained from \( N \) writing its digits in reverse order. The operation of taking the reversal is dependent on the base. In the definition of a \( b \)-ARH-number/h-ARH number \( N \) we take the reversal of the base-\( b \) representation of \( s_b(N)M \).

**Theorem 2.** Let \( \alpha \geq 1 \) integer, \( b \geq \alpha + 1 \) integer, and \( k = (1+\alpha)\ell, \ell \geq 0 \). Assume \( b \equiv 2+\alpha \mod (2+2\alpha) \). Define

\[
N_k = [(1\alpha)^\wedge k]_b.
\]

Then there exists \( M \geq 0 \) integer such that

\[
s_b(N_k) \cdot M = (s_b(N_k) \cdot M)^R = \frac{N_k}{2}.
\]

In particular, the numbers \( N_k, k \geq 1 \), are \( b \)-ARH numbers and \( b \)-Niven numbers.

The proof of Theorem 2 is done in Section 3.

**Remark 3.** The particular case \( b = 10, \alpha = 2 \), of Theorem 2, which gives \( N_k = (12)^\wedge 3^\ell \), is covered by [6, Example 10]. Theorem 2 does not give any information if \( b = 2 \).

The following proposition gives positive answers to [6, Questions 5 and 6].

**Proposition 4.** For any \( b \geq 2 \), there exist infinitely many \( b \)-ARH numbers and infinitely many additive multipliers. The \( b \)-ARH numbers are also \( b \)-Niven numbers.

The proof of Proposition 4 is done in Section 4.

**Remark 5.** Note that [6, Theorems 12 and 15] show, for all even bases, infinitely many \( b \)-ARH numbers that are not \( b \)-Niven numbers. The case of odd base is open. The question of finding infinitely many \( b \)-Niven numbers that are not \( b \)-ARH numbers is also open. It is shown in [6, Theorem 28] that for any base there exist infinitely many numbers that are not \( b \)-ARH numbers.

The result in Theorem 2 gives many base-10 solutions for the equation:

\[
N \cdot M = (N \cdot M)^R. \tag{1}
\]

One can try to solve the equation (1), where \((N \cdot M)^R\) is the reversal of \( N \cdot M \) written in base \( b \), for any numeration base \( b \).

Observe that if \( N \) is divisible by \( b \), then \((N \cdot M)^R\) has less digits then \( N \cdot M \), therefore \( N \) is not a solution of (1). Note also that if \( N = N^R \) and \( N \) has \( k \) digits then (1) always has an infinite set of solutions with

\[
M = [(1(0)^\wedge \ell)^\wedge p]_b, \ell \geq k - 1, p \geq 0.
\]

Consequently, if \((N_0, M_0)\) is a solution of (1), then (1) has infinite sets of solutions of types \((N_0, M)\) and \((N, M_0)\).
Theorem 6. Let $b \geq 2$ and $N \geq 1$ integer such that $b \nmid N$. Then $N$ is a solution of (1).

The proof of Theorem 6 is done in Section 5. For base 10, a proof belonging to David Radcliffe can be found at [8]. We learned about this reference from J. Shallit. We generalize the proof for an arbitrary numeration base. After our paper was written, we learned from J. Shallit [9] that he also has a proof of Theorem 6.

A $b$-numeric palindrome is a base-$b$ integer $N$ such that $N = N^R$.

Corollary 7. All integers, not divisible by $b$, are factors of $b$-numeric palindromes.

Definition 8. The multiplicity of a multiplicative multiplier $M$ is the number of $(N, M)$ solutions of (1).

It was observed above that for any solution $(N, M)$ of (1), $M$ has infinite multiplicity. The following theorem shows infinitely many solutions of (1) independent of above.

Theorem 9. Let $b \geq 2$ a numeration base. Then, for all $k \geq 0$, we have

$$[1(b - 1)]_b \cdot [(b - 1)^k]_b = [1(b - 2)(b - 1)^{k-2}(b - 2)]_b.$$  

The proof of Theorem 9 is done in Section 6.

Our next results show, for $b$ even, more examples of infinite sets of of $b$-ARH.

Theorem 10. Let $b \geq 2$ even. Let $a \in \{1, 2, \ldots, b - 1\}$ and let $k \geq 0$ be an integer.

(a) Let

$$N_k = [a(0)^k a]_b.$$  

Then $N_k$ is a $b$-ARH number, but not a $b$-Niven number.

(b) Let

$$N_k = [(10)^k 0 (10)^k]_b.$$  

Then $N_k$ is a $b$-ARH number, but not a $b$-Niven number.

(c) Let

$$N_k = [(0)^k 1^k 0 (0)^k]_b.$$  

Then $N_k$ is a $b$-ARH number and a $b$-Niven number.

The proof of Theorem 10 is done in Section 7.

The following theorem gives partial answers to [6, Questions 7 and 8].

Theorem 11.

(a) Let $b \equiv 4 \pmod{6}$. Let $k \geq 1$ integer such that $k \equiv 1 \pmod{3}$. Define $\alpha_k = [10^k(b - 2)]_b$. 

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Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a $b$-MRH number.

(b) Let $b = 2$ and let $k \geq 1$ be an even integer. Define

$$\alpha_k = [1(0)^k 1]_2.$$ 

Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a $b$-MRH number.

In particular, for any numeration base $b$, $b \equiv 4 \pmod{6}$, and for $b = 2$, there exist infinitely many $b$-MRH numbers and infinitely many multipliers.

The proof of Theorem 11 is done in Section 8.

Our next result lists several infinite sequences of 10-MRH numbers.

**Proposition 12.** Assume $k \geq 1$ integer and define $N_k = \alpha_k \cdot (\alpha_k)^R$, where $\alpha_k$ is one of the following numbers:

- $[1(0)^k 8]_{10}, k \equiv 1 \pmod{3}$,
- $[7(0)^k 2]_{10}$,
- $[5(0)^k 4]_{10}$,
- $[4(0)^k 5]_{10}$

Then $N_k$ is a 10-MRH number.

The first item in Proposition 12 follows as a corollary of Theorem 11. The other items can be proved using the same approach as in the proof of Theorem 11.

If $b$ even, $b \equiv 0 \pmod{3}$ or $b \equiv 2 \pmod{3}$, the next theorem shows there are infinitely many numeration bases for which there exist infinitely many $b$-MRH numbers and infinitely many multipliers.

**Theorem 13.**

(a) Let $b \geq 18$, $b = 6a$, and $a \equiv 1 \pmod{25}$. Let $\alpha_k = [1(0)^k 4]_b$ with $k \equiv 4 \pmod{5}$. Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a $b$-MRH number. The corresponding multipliers are distinct.

(b) Let $b \geq 18$, $b = 8a$, $a \equiv 1 \pmod{25}$, and $a \equiv 1 \pmod{3}$. Let $\alpha_k = [1(0)^k 4]_b$ with $k \equiv 4 \pmod{20}$. Then $N_k = \alpha_k \cdot (\alpha_k)^R$ is a $b$-MRH number. The corresponding multipliers are distinct.

The proof of Theorem 13 is done in section 9.

**Theorem 14.** For any base $b$ and for any string of base $b$ digits $I$ there exist infinitely many $b$-Niven numbers that contain the string $I$ in their base-$b$ representation.

**Proof.** Let $I$ be a string of base-$b$ digits. There exist infinitely many base-$b$ strings $J$ such that $s_b([IJ]_b)$ is a power of $b$, say $b^k, k \geq 1$. Then the number $N_J = [IJ(0)^k]_b$ is a $b$-Niven number. \qed
3 Proof of Theorem 2

Proof. The condition $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$ implies that $b + \alpha$ is even. The base-$b$ representation for $N_k/2$ is $N_k/2 = \left(\left(\frac{b+\alpha}{2}\right)^\ell\right)_b$. One has that:

$$s_b(N_k) = k \cdot (1 + \alpha) = (1 + \alpha)^{\ell+1}. \quad (2)$$

The value of $N_k/2$ in base 10 is obtained summing a geometric series.

$$\frac{N_k}{2} = \frac{b + \alpha}{2} \cdot \frac{b^{2\ell-2} + b + \alpha}{2} \cdot \frac{b^{2\ell-4} + \cdots + b + \alpha}{2} \cdot \frac{b^2 + b + \alpha}{2} = \frac{b + \alpha}{2} \cdot \frac{b^{2\ell} - 1}{b^2 - 1}. \quad (3)$$

Note that $N_k/2 = (N_k/2)^R$. We finish the proof of the theorem if we show that:

$$(1 + \alpha)^{\ell+1} \mid \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell} - 1}{b^2 - 1}. \quad (4)$$

We prove (4) by induction on $\ell$. For $\ell = 0$ equation (4) becomes $1 + \alpha \mid \frac{b+\alpha}{2}$, which is true because $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$.

Now we assume that (4) is true for $\ell$ and show that it is true for $\ell + 1$.

$$\frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell} - 1}{b^2 - 1} = \frac{b + \alpha}{2} \cdot \frac{\left(\frac{b^{2(1+\alpha)^\ell}}{b^2 - 1}\right)^{1+\alpha} - 1}{b^2 - 1} \quad (5)$$

$$= \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)^\ell} - 1}{b^2 - 1} \left(B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1\right),$$

where

$$B = b^{2(1+\alpha)^\ell}. \quad (6)$$

The congruence $b \equiv 2 + \alpha \pmod{2 + 2\alpha}$ implies that

$$b^2 \equiv (2 + \alpha)^2 \equiv \alpha^2 + 4\alpha + 4 \equiv \alpha^2 \equiv 1 \pmod{1 + \alpha},$$

which implies that

$$b^m \equiv 1 \pmod{1 + \alpha}, \quad m \text{ even.} \quad (7)$$

From (6) and (7) follows that $B^p \equiv 1 \pmod{1 + \alpha}, 1 \leq p \leq \alpha$, so

$$1 + \alpha \mid B^\alpha + B^{\alpha-1} + \cdots + B^2 + B + 1. \quad (8)$$

Combining (4) (for $\ell$) and (8), and using (5), it follows that (4) is true for $\ell + 1$. \qed
4 Proof of Proposition 4

Proof. The case $b = 2$ is covered by [6, Theorem 12]. If $b \geq 3$, choose $\alpha = b - 2$ and apply Theorem 2. We show that, for a fixed $b$, the multipliers appearing in the proof of Theorem 2 are all distinct. It follows from (2) and (3) that the multiplier for $N_k$ is given by:

$$M = \frac{N_k}{s_b(N_k)} = \frac{b + \alpha}{2} \cdot \frac{b^{2(1+\alpha)\ell}-1}{b^\ell-1}. \quad (9)$$

Note that $\alpha = b - 2$. After algebraic manipulations, equation (9) becomes

$$M = \frac{b^{2(1+\alpha)\ell}-1}{(b - 1)^\ell(b^2 - 1)}.$$ 

In order to show that the multipliers are distinct it is enough to show that the sequence of multipliers is strictly increasing as a function of $\ell$. That is, we need to show that:

$$\frac{b^{2(1+\alpha)\ell}-1}{(b - 1)^\ell(b^2 - 1)} < \frac{b^{2(1+\alpha)\ell+1}-1}{(b - 1)^{\ell+1}(b^2 - 1)}. \quad (10)$$

After algebraic manipulations (10) becomes

$$(b - 1)(b^{2(1+\alpha)\ell} - 1) < b^{2(1+\alpha)\ell+1} - 1. \quad (11)$$

After denoting

$$B = b^{2(1+\alpha)\ell} = b^{2(b-1)\ell},$$

right hand side of (11) factors as follows:

$$b^{2(1+\alpha)\ell+1} - 1 = (b^{2(1+\alpha)\ell} - 1)(B^\alpha + B^{\alpha-1} + \cdots + B + 1). \quad (12)$$

Now (11) follows from (12) and the following inequality:

$$b - 1 < b^{2(b-1)\ell} \quad \ell \geq 0, \ell \geq 0, b \geq 3.$$ 

5 Proof of Theorem 6

Proof. Let $b = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\alpha_i \geq 1$, $p_i$ prime, $1 \leq i \leq k$. We recall that a base-$b$ integer $N$ is divisible by $p_i^\gamma$ if the last $\gamma$ digits of $N$ form a base-$b$ integer divisible by $p_i^\gamma$. Let $N = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} w$, where $\text{gcd}(w, b) = 1$. Let $m = \max(\beta_1, \beta_2, \cdots, \beta_k)$. Let $L$ be the base-$b$ integer equal to $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$. As $b \nmid N$, the last digit of $L$ is not 0. Let $\ell$ be the length of
L. Consider the base-\(b\) palindrome \(P = [L^R(0)^{m-\ell}L]_b\), where \(L^R\) is the reversal of base-\(b\) representation of \(L\). As \(P\) is divisible by \(p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}\), this is the end of the proof if \(w = 1\).

Assume \(w > 1\). Let \(\phi\) be Euler’s totient function which counts the positive integers up to a given integer \(n\) that are relatively prime to \(n\). As \(\gcd(w, b) = 1\) Euler’s theorem implies that \(b^{\phi(w)} - 1 \equiv 0 \pmod{w}\).

Let \(r\) be an multiple of \(\phi(w)\) which is greater than \(l + m\), the length of \(P\). Let \(q \geq 1\) a multiple of \(b^{\phi(w)} - 1\). Consider the infinite family of integers given by

\[
Q_{r,q} = [1(0)^{r-1}1]_b = 1 + b^r + b^{2r} + \cdots + b^{qr} = 1 + b^r + b^{2r} + \cdots + b^{qr} + q - q
\]

Let \(Q_{r,q}\) be divisible by \(b^{\phi(w)} - 1\) and by \(w\). We finish the proof observing that \(P \cdot Q_{r,q}\) is a base-\(b\) palindrome divisible by \(N\). \(\square\)

6 Proof of Theorem 9

Proof. Observe that:

\[
(b - 1) \cdot (b - 1) = b(b - 2) + 1 = [(b - 2)1]_b
\]

\[
(b - 1)b^k + (b - 1)b^k = b^k + (b - 2)b^{k-1} = [1(b - 2)0^\wedge k]_b.
\] (14)

Using (14) we get

\[
[1(b - 1)]_b \cdot [(b - 1)^\wedge k]_b = (b + b - 1) \cdot \left(\sum_{i=0}^{k-1} (b - 1)b^i\right)
\]

\[
= \sum_{i=0}^{k-1} \left((b - 1)b^{i+1} + (b(b - 2) + 1) b^i\right)
\]

\[
= \sum_{i=1}^{k} (b - 1)b^i + \sum_{i=0}^{k-1} (b(b - 2) + 1) b^i
\]

\[
= (b - 1)b^k + \sum_{i=1}^{k-1} (b - 1)b^{i+1} + (b(b - 2) + 1)
\]

\[
= (b - 1)b^k + (b - 1)b^k + \sum_{i=1}^{k-2} (b - 1)b^{i+1} + (b(b - 2) + 1)
\]

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\[ = b^k + (b - 2)b^{k-1} + \sum_{i=1}^{k-2} (b - 1)b^{i+1} + b(b - 2) + 1 \]
\[ = [1(b - 2)(b - 1)^{k-2}(b - 2)1]_b. \]

7 Proof of Theorem 10

Proof. (a) Note that \( s_b(N_k) = 2a \). As \( b \) is even, there exists an integer \( M \) such that:
\[ 2a \cdot M = [a(0)^{k+1}]_b. \]

The following computation shows that \( N_k \) is a \( b \)-ARH number:
\[ s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R = [a(0)^{k+1}]_b + [a]_b = [a(0)^k a]_b = N_k. \]

To show that \( N_k \) is not \( b \)-Niven observe that \( N_k/a = [1(0)^{k+1}]_b \) is odd.

(b) Note that \( s_b(N_k) = 2b \). As \( b \) is even, the multiplier \( M = [(1(0)^k)^b(0)^{kb+b-1}]_b/2 \) is an integer.

The following computation shows that \( N_k \) is a \( b \)-ARH number:
\[ s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R \]
\[ = [(1(0)^k)^{b(0)^{kb+b}}]_b + [(0)^{k+1}]_b = [(1(0)^k)^{b0((0)^k1)^{b}}]_b = N_k. \]

To show that \( N_k \) is not \( b \)-Niven observe that \( N_k \) is not divisible by \( b \).

(c) The proof is similar to that of b).

8 Proof of Theorem 11

Proof. (a) Using the fact that
\[ (b - 2)^2 = b^2 - 4b + 4 = b(b - 4) + 4 = [(b - 4)4]_b, \]
an equivalent base-\( b \) representation for \( N_k \) is given by
\[ N_k = \begin{cases} 
[(b - 2)(0)^{k-1}(b - 4)5(0)^k(b - 2)]_b, & \text{if } b \neq 4; \\
[2(0)^{k-1}11(0)^k2]_b, & \text{if } b = 4. 
\end{cases} \tag{15} \]

If \( b \neq 4 \) one has \( s_b(N_k) = 3(b - 1) \) and if \( b = 4 \) one has \( s_4(N_k) = 6 \). To finish the proof of case a) it is enough to show that \( \alpha_k \) is divisible by \( s_b(N_k) \).
If $b \neq 4$ we get
\[ \alpha_k = b^{k+1} + b - 2 = b^{k+1} - 1 + b - 1 = (b - 1)(b^k + b^{k-1} + \cdots + b^2 + b + 2) \]
and
\[ b^k + b^{k-1} + \cdots + b^2 + b + 2 \equiv k + 2 \equiv 0 \pmod{3}. \]
For the first congruence we used $b \equiv 1 \pmod{3}$ and for the second we used $k \equiv 1 \pmod{3}$.

If $b = 4$, then clearly $\alpha_k$ is divisible by 2. Moreover
\[ \alpha_k = 4^{k+1} + 2 = (3 + 1)^{k+1} + 2 \equiv 0 \pmod{3}, \]
which shows that $\alpha_k$ is divisible by 6.

(b) Now assume that $b = 2$. Then an equivalent base-2 representation for $N_k$ is given by
\[ N_k = \lfloor 1(0)^{k-1}10(0)^k1 \rfloor_2, \]
so $s_2(N_k) = 3$. To finish the proof, we use the fact that $k$ is even to show that $\alpha_k$ is divisible by 3:
\[ \alpha_k = 2^{k+1} + 1 = (3 - 1)^{k+1} + 1 \equiv 0 \pmod{3}. \]

To prove the last claim in the theorem, we show that the multipliers corresponding to various values of $k$ are distinct. This follows from the explicit formulas below. All sequences of multipliers are strictly increasing as functions of $k$.

If $b = 2$ the sequence of multipliers is given by $M_k = \frac{2^{k+1} + 1}{3}$. If $b = 4$ the sequence of multipliers is given by $M_k = \frac{4^{k+1} + 2}{6}$. If $b > 4$ the sequence of multipliers is given by $M_k = \frac{b^{k+1} + b - 2}{3(b-1)}$.

\[ \square \]

9 Proof of Theorem 13

Proof. (a) The base-$b$ representation for $N_k$ is
\[ N_k = \lfloor 4(0)^{k-1}(17)(0)^k4 \rfloor_b. \]
Therefore $s_b(N_k) = 25$. If $k = 5\ell + 4$, one has that:
\[ \alpha_k = 6^k a^k + 4 \equiv (6^5)^\ell 6^4 + 4 \equiv (7776)^\ell \cdot 296 + 4 \equiv 0 \pmod{25}. \]
Hence $N_k$ is a $b$-MRH number with multiplier $\frac{\alpha_k}{25} = \frac{(6a)^{k+4}}{25}$.

(b) As above, $s_b(N_k) = 25$. If $k = 20\ell + 4$, one has that:
\[ \alpha_k = 8^k a^k + 4 \equiv (8^{20})^\ell 8^4 + 4 \equiv (76)^\ell \cdot 96 + 4 \equiv 0 \pmod{25}. \]
Hence $N_k$ is a $b$-MRH number with multiplier $\frac{\alpha_k}{25} = \frac{(8a)^{k+4}}{25}$. \[ \square \]
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