A Graph-Theoretic Model for a Generalized Fibonacci Gem

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Abstract
We extend a charming Fibonacci pleasantry to Fibonacci, Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials; and then confirm the resulting polynomial delights using graph-theoretic tools.

1 Introduction

Generalized Fibonacci polynomials $z_n(x)$ are defined by the recurrence $z_n(x) = a(x)z_{n-1}(x) + b(x)z_{n-2}(x)$, where $x$ is an arbitrary complex variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 2$.

Let $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th Fibonacci polynomial; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th Lucas polynomial. Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [2, 3, 12, 13].

Pell polynomials $p_n(x)$ and Pell-Lucas polynomials $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The Pell numbers $P_n$ and Pell-Lucas numbers $Q_n$ are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7, 10].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the $n$th Jacobsthal polynomial; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the $n$th Jacobsthal-Lucas polynomial [5, 6]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the $n$th Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$. 

The polynomials \( f_n(x), l_n(x), J_n(x), \) and \( j_n(x) \) can also be defined explicitly using Binet-like formulas:

\[
\begin{align*}
  f_n(x) &= \frac{\alpha^n - \beta^n}{\alpha - \beta}; &
  l_n(x) &= \alpha^n + \beta^n; \\
  J_n(x) &= \frac{u^n - v^n}{u - v}; &
  j_n(x) &= u^n + v^n,
\end{align*}
\]

where \( \alpha = \alpha(x) \) and \( \beta = \beta(x) \) are the solutions of the equation \( t^2 - xt - 1 \), and \( u = u(x) \) and \( v = v(x) \) are those of \( t^2 - t - x = 0 \). Notice that \( \alpha - \beta = \sqrt{x^2 + 4} \) and \( u - v = \sqrt{4x + 1} \).

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so \( z_n \) means \( z_n(x) \). In addition, we let \( g_n = f_n \) or \( l_n \); \( b_n = p_n \) or \( q_n \); and \( c_n = J_n \) or \( j_n \); and correspondingly, \( G_n = F_n \) or \( L_n \); \( B_n = P_n \) or \( Q_n \); and \( C_n = J_n \) or \( j_n \).

2 \( Q \)-matrix and digraph

Gibonacci polynomials \( f_n \) and \( l_n \) can be studied using the \( Q \)-matrix

\[
Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},
\]

where \( Q = Q(x) = (q_{ij})_{2 \times 2} \) \([11, 14]\). It then follows by induction that

\[
Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},
\]

where \( n \geq 1 \).

The \( Q \)-matrix has a graph-theoretic appeal. It can be interpreted as the weighted adjacency matrix of a weighted digraph \( D_1 \) with vertices \( v_1 \) and \( v_2 \) \([11, 14]\); see Figure 1. Notice that a weight is assigned to each edge.

A walk from vertex \( v_i \) to vertex \( v_j \) is a sequence \( v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j \) of vertices \( v_k \) and edges \( e_k \), where edge \( e_k \) is incident with vertices \( v_k \) and \( v_{k+1} \). The walk is closed if \( v_i = v_j \); otherwise, it is open. The length of a walk is the number of edges in the walk. The weight of a walk is the product of the weights of the edges along the walk.

We can employ the weighted adjacency matrix to compute the weight of a walk of length \( n \) from any vertex \( v_i \) to any vertex \( v_j \), as the following theorem shows \([9, 11]\).
Theorem 1. Let $A$ be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the $ij$th entry of the matrix $A^n$ gives the sum of the weights of all walks of length $n$ from $v_i$ to $v_j$, where $n \geq 1$.

This theorem implies the following result.

Corollary 2. The $ij$th entry of $Q^n$ gives the sum of the weights of all walks of length $n$ from $v_i$ to $v_j$ in the weighted digraph $D_1$, where $1 \leq i, j \leq 2$.

It follows by this corollary that the sum of the weights of all closed walks of length $n$ originating in the digraph model is $f_{n+1}$, and that of walks of length $n$ originating at $v_2$ is $f_{n-1}$. So the sum of the weights of all closed walks of length $n$ is $f_{n+1} + f_{n-1} - l_n$.

3 A Gibonacci delight

In 1963, H. W. Gould established a charming identity for Fibonacci squares [8, 13]:

$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2.$$  \hspace{1cm} (1)

It has a simple, but delightful geometric interpretation [13].

The next theorem extends identity (1) to gibonacci polynomials $g_n$.

Theorem 3.

$$g_{n+3}^2 = f_3g_{n+2}^2 + f_3g_{n+1}^2 - g_n^2.$$  \hspace{1cm} (2)

Proof. Using the gibonacci recurrence, we have

$$g_{n+3}^2 + g_n^2 = (xg_{n+2} + g_{n+1})^2 + (g_{n+2} - xg_{n+1})^2$$
$$= (x^2 + 1)g_{n+2}^2 + (x^2 + 1)g_{n+1}^2.$$

This yields the desired identity. (It also follows by Binet’s formulas.) \hfill $\Box$
4 Graph-theoretic models

With these tools at our finger tips, we can give graph-theoretic interpretations of the gi-bonacci results in Theorem 3. The essence of our technique hinges on Corollary 2, and the “weighted” version of Fubini’s principle [1, 13]: Counting the number of elements in a set in two different ways yields the same result.

We begin our discourse with $g_n = f_n$.

4.1 Interpretation with $g_n = f_n$

It follows by Corollary 2 that the sum of the weights of closed walks of length $n+2$ originating at $v_1$ is $f_{n+3}$. The sum $S$ of the weights of ordered pairs $(v, w)$ of such closed walks is the product of the sum of the weights of such walks $v$ and $w$. Consequently, $S = f_{n+3}^2$.

We will now compute the sum $S$ in a different way.

**Proof.** Case 1. Suppose $v$ and $w$ begin with a loop at $v_1$. The sum of the weights of pairs $(v, w)$ of such closed walks of length $n+2$ is $(xf_{n+2})(xf_{n+2}) = x^2 f_{n+2}^2$.

Case 2. Suppose $v$ begins with a loop at $v_1$, but $w$ does not. The sum of the weights of pairs of such closed walks is $(xf_{n+2})(1 \cdot 1 \cdot f_n) = xf_{n+2}f_n$.

Case 3. On the other hand, suppose $v$ does not begin with a loop, but $w$ does. The sum of the weights of pairs of such closed walks is $(1 \cdot 1 \cdot f_n)(xf_{n+2}) = xf_{n+2}f_n$.

Case 4. Finally, suppose neither $v$ nor $w$ begins with a loop. The contribution of pairs of such walks toward the sum $S$ is $(1 \cdot f_{n+1})(1 \cdot f_{n+1}) = f_{n+1}^2$.

Combining the four cases, we also get

$$S = x^2 f_{n+2}^2 + f_{n+1}^2 + 2xf_{n+2}f_n$$

$$= (x^2 + 1)f_{n+2}^2 + (x^2 + 1)f_{n+1}^2 - f_n^2,$$

as in the proof of Theorem 3.

Equating the cumulative sums yields the desired result. 

As a byproduct, this discourse then gives a graph-theoretic proof of the Pell identity

$$p_{n+3}^2 = p_{3n+2}^2 + p_{3n+1}^2 - p_n^2.$$

Next we investigate the graph-theoretic interpretation of identity (2) with $g_n = l_n$. 

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4.2 Interpretation with $g_n = l_n$

Proof. Let $A$ denote the set of closed walks of length $n + 3$ originating at $v_1$, and $B$ that of length $n + 3$ originating at $v_2$. Let $C = A \cup B$, where $A \cap B = \emptyset$. The sum of the weights of all closed walks in $C$ equals $f_{n+4} + f_{n+2} = l_{n+3}$. Consequently, the sum $S$ of the weights of ordered pairs $(v, w) \in C \times C$ is given by $S = l_{n+3}^2$.

To compute this sum in a different way, first we make an interesting observation. By Theorem 3, we have

$$x^2f_{n+3}^2 + 4f_{n+2}^2 + 4xf_{n+3}f_{n+2} = (xf_{n+3} + 2f_{n+2})^2 = (f_{n+4} + f_{n+2})^2 = l_{n+3}^2$$

Consequently, it suffices to establish graph-theoretically the equivalent identity

$$x^2f_{n+3}^2 + 4f_{n+2}^2 + 4xf_{n+3}f_{n+2} = l_{n+3}^2.$$  \hspace{1cm} (4)

We will accomplish this using four cases for an arbitrary element $(v, w) \in C \times C$.

**Case 1.** Suppose $v, w \in A$. Suppose both $v$ and $w$ begin with a loop. The sum of the weights of pairs $(v, w)$ of such closed walks is $(xf_{n+3})(xf_{n+3}) = x^2f_{n+3}^2$. If $v$ begins with a loop at $v_1$ and $w$ does not, then $v \in A$ and $w \in B$. The sum of the weights of all such pairs $(v, w)$ of closed walks equals $(x \cdot f_{n+3})(1 \cdot 1 \cdot f_{n+2}) = xf_{n+3}f_{n+2}$. Suppose $v$ does not begin with a loop, but $w$ does. Then $v \in B$ and $w \in A$. The sum of the weights of all such pairs $(v, w)$ of closed walks equals $(1 \cdot 1 \cdot f_{n+2})(x \cdot f_{n+3}) = xf_{n+3}f_{n+2}$. Suppose neither $v$ nor $w$ begins with a loop. The total contribution by the corresponding pairs $(v, w)$ is $(1 \cdot 1 \cdot f_{n+2})(1 \cdot 1 \cdot f_{n+2}) = f_{n+2}^2$.

Thus, when $v, w \in A$, the sum of the weights of such closed walks of length $n + 3$ is given by

$$S_1 = x^2f_{n+3}^2 + 2xf_{n+3}f_{n+2} + f_{n+2}^2.$$

**Case 2.** Suppose $v \in A$ and $w \in B$. If $v$ begins with a loop, then the sum of the weights of products of such closed walks of length $n + 3$ is $(xf_{n+3})(f_{n+2}) = xf_{n+3}f_{n+2}$. On the other hand, suppose $v$ does not begin with a loop. The corresponding sum is $(1 \cdot 1 \cdot f_{n+2})(f_{n+2}) = f_{n+2}^2$. Consequently, the total contribution from this case is

$$S_2 = xf_{n+3}f_{n+2} + f_{n+2}^2.$$

**Case 3.** Suppose $v \notin A$, but $w \in B$. Then $v \in B$. If $w$ begins with a loop, the resulting contribution is $(f_{n+2})(xf_{n+3}) = xf_{n+3}f_{n+2}$. If $w$ does not begin with a loop, then the corresponding contribution is $(f_{n+2})(1 \cdot 1 \cdot f_{n+2}) = f_{n+2}^2$. Consequently, the total contribution from Case 3 toward the cumulative sum is

$$S_3 = xf_{n+3}f_{n+2} + f_{n+2}^2.$$
Case 4. Suppose \( v, w \in B \). Clearly, the resulting contribution from this case toward \( S \) is

\[
S_4 = (f_{n+2})(f_{n+2}) = f_{n+2}^2.
\]

Collecting all contributions from the four cases and using identities (2) and (3), we get

\[
S = S_1 + S_2 + S_3 + S_4 = x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4xf_{n+3}f_{n+2} = l_{n+3}^2,
\]

as desired.

An Alternate Proof.

Proof. Alternatively, by focusing on the closed walks at \( v_1 \) alone, we can establish identity (3). To see this, let \( C \) denote the set of closed walks of length \( n + 3 \) at \( v_1 \), and \( D \) that of length \( n + 1 \) at \( v_1 \). Let \( E = C \cup D \), where \( C \cap D = \emptyset \). The sum of the weights of the walks in \( E \) is \( f_{n+4} + f_{n+2} = l_{n+3} \). Consequently, the sum \( S \) of the weights of elements in \( E \times E \) is \( S = l_{n+3}^2 \).

We will now compute \( S \) in a different way. (In the interest of brevity, we highlight the key steps only.) To this end, let \((v, w)\) be an arbitrary element in \( E \times E \).

Suppose \( v, w \in C \). Then the sum of the weights of the pairs \((v, w)\) of such closed walks is given by

\[
S_1 = x^2 f_{n+3}^2 + f_{n+2}^2 + 2xf_{n+3}f_{n+2}.
\]

On the other hand, let \( v \in C \) and \( w \in D \). The total contribution from such pairs \((v, w)\) is

\[
S_2 = x^2 f_{n+3}^2 f_{n+1} + xf_{n+3} f_n + xf_{n+2} f_{n+1} + f_{n+2} f_n
= f_{n+2}^2 + xf_{n+3} f_{n+2}.
\]

When \( v, w \in D \), the total contribution from the corresponding pairs is

\[
S_3 = x^2 f_{n+1}^2 + 2xf_{n+1} f_n + f_n^2
= f_{n+2}^2.
\]

Finally, let \( v \in D \) and \( w \in C \). The corresponding contribution is

\[
S_4 = x^2 f_{n+3} f_{n+1} + xf_{n+3} f_n + xf_{n+2} f_{n+1} + f_{n+2} f_n
= f_{n+2}^2 + xf_{n+3} f_{n+2}.
\]

Thus the cumulative sum \( S \) of the weights of all pairs \((v, w)\) in \( E \times E \) is also given by

\[
S_1 + S_2 + S_3 + S_4 = x^2 f_{n+3}^2 + 4f_{n+2}^2 + 4xf_{n+3} f_{n+2} = l_{n+3}^2,
\]

as expected.
Since [4, 14]
\[ g_{n+1}^2 + g_n^2 = \begin{cases} f_{2n+1}, & \text{if } g_n = f_n; \\ (x^2 + 4)f_{2n+1}, & \text{if } g_n = l_n; \end{cases} \]
these models also give a graph-theoretic interpretation of the identity [2, 4, 14]
\[ g_{n+3}^2 + g_n^2 = (x^2 + 1)(g_{n+2}^2 + g_{n+1}^2) \]
\[ = \begin{cases} (x^2 + 1)f_{2n+3}, & \text{if } g_n = f_n; \\ (x^2 + 1)(x^2 + 4)f_{2n+3}, & \text{if } g_n = l_n. \end{cases} \]

We now add that using the bijection algorithm in [11], we can translate the graph-theoretic models into tiling models with squares and dominoes, where weight(square) = \( x \); weight(domino) = 1; and the weight of a tiling is the product of the weights of tiles in the tiling.

Replacing \( x \) with \( 2x \) in this discourse yields a graph-theoretic proof of the Pell-Lucas identity
\[ q_{n+3}^2 = 4x^2p_{n+3}^2 + 4p_{n+2}^2 + 8xp_{n+3}p_{n+2} \]
\[ = p_3q_{n+2}^2 + p_3q_{n+1}^2 - q_n^2. \]

Finally, it follows from identity (4) that
\[ F_{n+3}^2 + 4F_{n+2}^2 + 4F_{n+3}F_{n+2} = L_{n+3}^2. \]
Consequently, an \( L_{n+3} \times L_{n+3} \) floor can be tessallated with nine tiles: one \( F_{n+3} \times F_{n+3} \) tile; four \( F_{n+2} \times F_{n+2} \) tiles; and four \( F_{n+3} \times F_{n+2} \) tiles, where \( n \geq 0 \).

5 Jacobsthal implications

Using the gibbonacci-Jacobsthal relationships \( J_n(x) = x^{(n-1)/2}f_n(u) \) and \( j_n(x) = x^{n/2}l_n(u) \) [12], we can easily find the Jacobsthal counterparts of identities (2) and (3), where \( u = 1/\sqrt{x} \):
\[ c_{n+3}^2 = J_3(x)c_{n+2}^2 + xJ_3(x)c_{n+1}^2 - x^3c_n^2; \]
\[ j_{n+1}^2(x) = J_{n+1}^2(x) + 4x^2J_n^2(x) + 4xJ_{n+1}(x)J_n(x), \]
respectively. (We have omitted the basic algebra for brevity and convenience.) Consequently,
\[ C_{n+3}^2 = 3C_{n+2}^2 + 6C_{n+1}^2 - 8C_n^2; \]
\[ j_{n+1}^2 = J_{n+1}^2 + 16J_n^2 + 8J_{n+1}J_n. \]
Identity (6) implies that a \( j_{n+1} \times j_{n+1} \) floor can be tiled with 25 tiles: one \( J_{n+1} \times J_{n+1} \) tile; sixteen \( J_n \times J_n \) tiles; and eight \( J_{n+1} \times J_n \) tiles, where \( n \geq 1 \).

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5.1 A Jacobsthal digraph

Next we confirm independently identity (5) using graph-theoretic tools. To this end, we first present a weighted digraph $D_2$; see Figure 2. Its weighted adjacency matrix is

$$M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}.$$ 

Then

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$; we can confirm this using induction.

![Figure 2: Weighted digraph $D_2$](image)

It then follows that the sum of the weights of closed walks of length $n$ originating at $v_1$ is $J_{n+1}(x)$, and that of those originating at $v_2$ is $xJ_{n-1}(x)$. Consequently, the sum of all closed walks of length in the digraph $D_2$ is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$. These facts play a central role in the graph-theoretic proof.

With these tools at our finger tips, we now present the proof of each part.

**Proof.**

**Part 1.** To establish part 1, we let $A$ be the set of closed walks of length $n+2$ starting at $v_1$. The sum of the weights of all such closed walks is $J_{n+3}(x)$; so the sum $S$ of the weights of all ordered pairs $(v, w) \in A \times A$ is $J_{n+3}^2(x)$.

We will now compute $S$ in a different way. Again, let $(v, w)$ be an arbitrary element of $A \times A$. Suppose both $v$ and $w$ begin with a loop; the sum of the weights of such pairs $(v, w)$ is $[1 \cdot 1 \cdot J_{n+2}(x)][1 \cdot 1 \cdot J_{n+2}(x)] = J_{n+2}^2(x)$. If $v$ begins with a loop and $w$ does not, the corresponding sum is $[1 \cdot J_{n+2}(x)][1 \cdot J_{n+1}(x)] = xJ_{n+2}(x)J_{n+1}(x)$. Suppose $v$ does not begin with a loop, but $w$ does; then also the resulting sum is $[x \cdot 1 \cdot J_{n+1}(x)][1 \cdot J_{n+2}(x)] = xJ_{n+2}(x)J_{n+1}(x)$. 

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Finally, if both \( v \) and \( w \) do not begin with a loop, the contribution from such pairs equals 
\[
[x \cdot 1 \cdot J_{n+1}(x)][x \cdot 1 \cdot J_{n+1}(x)] = x^2 J_{n+1}^2(x).
\]

Thus the cumulative contribution of pairs \((v, w)\) all closed walks of length \( n+2 \) starting at \( v_1 \) is given by

\[
S = J_{n+2}^2(x) + 2xJ_{n+2}(x)J_{n+1}(x) + x^2 J_{n+1}^2(x)
= J_{n+2}^2(x) + xJ_{n+2}(x)[J_{n+2}(x) - xJ_{n}(x)] + xJ_{n+1}(x)[J_{n+1}(x) + xJ_{n}(x)] + x^2 J_{n+1}^2(x)
= (x + 1)J_{n+2}^2(x) + x(x + 1)J_{n+1}^2(x) - x^2J_{n}(x)[J_{n+2}(x) - J_{n+1}(x)]
= (x + 1)J_{n+2}^2(x) + x(x + 1)J_{n+1}^2(x) - x^3 J_{n}(x).
\]

Combining the two values of \( S \) yields identity (5) when \( c_n = J_n(x) \).

**Part 2.** To confirm identity (5) when \( c_n = j_n(x) \), we focus on the closed walks of lengths \( n + 3 \) and \( n \) in the digraph. Let \( C \) be the set of closed walks of length \( n + 3 \) starting at \( v_1 \), and \( D \) the set of those starting at \( v_2 \). Clearly, \( C \cap D = \emptyset \), so the sum of the weights of the walks in \( F = C \cup D \) is \( j_{n+3}(x) \). Consequently, the sum \( S_1 \) of the weights of the ordered pairs \((v, w)\) such pairs equals \( j_{n+3}^2(x) \).

Now let \( R \) denote the set of closed walks of length \( n \) originating at \( v_1 \), and \( S \) that of those originating at \( v_2 \). It follows by the preceding argument that the sum \( S_2 \) of the weights of the ordered pairs \((v, w)\) in \( G \times G \) is \( j_n^2(x) \), where \( G = R \cup S \) and \( R \cap S = \emptyset \).

Thus

\[
S_1 + x^3S_2 = j_{n+3}^2(x) + x^3 j_n^2(x).
\]

We will now compute the sum \( S_1 + x^3S_2 \) in a different way. Again, let \((v, w)\) be an arbitrary element of \( F \times F \).

Suppose \( v, w \in C \). Then the sum of the weights of pairs \((v, w)\) of such closed walks of length \( n + 3 \) originating at \( v_1 \) is \( [J_{n+4}(x)][J_{n+4}(x)] = J_{n+4}^2(x) \). If \( v \in C \) and \( w \in D \), then the resulting sum is \( [J_{n+4}(x)][xJ_{n+2}(x)] = xJ_{n+4}(x)J_{n+2}(x) \). When \( v \in D \) and \( w \in C \), the corresponding sum is \( [xJ_{n+2}(x)][J_{n+4}(x)] = xJ_{n+4}(x)J_{n+2}(x) \). Finally, when \( v, w \in D \), the contribution from such pairs \((v, w)\) is \([xJ_{n+2}(x)][xJ_{n+2}(x)] = x^2J_{n+2}^2(x) \). Thus

\[
S_1 = J_{n+4}^2(x) + 2xJ_{n+4}(x)J_{n+2}(x) + x^2 J_{n+2}^2(x).
\]

It then follows that

\[
S_2 = J_{n+1}^2(x) + 2xJ_{n+1}(x)J_{n}(x) + x^2 J_{n}^2(x).
\]

Consequently, \( S_1 + x^3S_2 = A + B \), where

\[
A = J_{n+4}^2(x) + x^2J_{n+2}^2(x) + x^3 J_{n+1}^2(x);
B = x^n J_{n-1}^2(x) + 2xJ_{n+4}(x)J_{n+2}(x) + 2x^4J_{n+1}(x)J_{n-1}(x).
\]
Proof. We will now confirm that $S_1 + x^3 S_2 = (x + 1)j_{n+2}^2(x) + x(x + 1)j_{n+1}^2(x)$. The proof involves a lot of carefully prepared basic algebra; so in the interest of brevity, clarity, and convenience, we present only the major steps; also we omit the argument in the functional notation.

We have

$$A = (J_{n+3} + xJ_{n+2})^2 + x^2 J_{n+2}^2 + x^3 J_{n+1}$$

$$= J_{n+3}^2 + 2x^2 J_{n+2}^2 + 2xJ_{n+2}(J_{n+2} + xJ_{n+1}) + x^3 J_{n+1}$$

$$= J_{n+3}^2 + (x^2 + x)J_{n+2}^2 + x^3 J_{n+1} + (x^2 + x)J_{n+2}^2 + 2x^2 J_{n+2}J_{n+1}$$

$$= J_{n+3}^2 + (x^2 + x)J_{n+2}^2 + x^3 J_{n+1} + (x^2 + x)J_{n+2}^2 + 2x^2 J_{n+1}(J_{n+3} - xJ_{n+1})$$

$$= J_{n+3}^2 + (x^2 + x)J_{n+2}^2 + x^3 J_{n+1} + (x^2 + x)J_{n+2}^2 + 2x^2 J_{n+3}J_{n+1} - 2x^3 J_{n+1}$$;

$$B = x^3(J_{n+1} - J_n)^2 + 2xJ_{n+2}(J_{n+3} + xJ_{n+2}) + 2x^3 J_{n+1}(J_{n+1} - J_n)$$

$$= x^3 J_{n+1}^2 + x^3 J_n^2 - 2x^3 J_{n+1}J_n + 2xJ_{n+3}J_{n+2} + 2x^2 J_{n+2}^2 + 2x^3 J_{n+1}^2 - 2x^3 J_{n+1}J_n$$

$$= x^3 J_{n+1}^2 + x^3 J_n^2 - 2x^3 J_{n+1}J_n + 2xJ_{n+3}J_{n+1} + x^3 J_{n+1}^2 + 2x^2 J_{n+2}^2 + 2x^3 J_{n+1}^2 - 2x^3 J_{n+1}J_n$$

$$= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^3 J_{n+1}J_n + 2x^2 J_{n+2}^2 + 3x^3 J_{n+1}^2 - 4x^3 J_{n+1}J_n$$

$$= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^2 J_{n+2} + xJ_{n+1} + x(J_{n+3} - J_{n+2})^2 + 2x^2 J_{n+2} + 2x^3 J_{n+1}^2 - 4x^3 J_{n+1}J_n$$

$$= 2xJ_{n+3}J_{n+1} + x^3 J_n^2 + 2x^2 J_{n+2} + xJ_{n+1} + xJ_{n+1}J_n$$

Then

$$S_1 + x^3 S_2 = C + D + (x^2 + x)J_{n+2}^2 - 2x^3 J_{n+1}J_n + (2x^2 + x)J_{n+2}^2 - 2xJ_{n+3}J_{n+2},$$

where

$$C = (x + 1)(J_{n+3}^2 + 2x J_{n+3}J_{n+1}) + x^3 J_{n+1}$$

$$= (x + 1)(J_{n+3} + x J_{n+1})^2 - x^2 J_{n+1}$$

$$= (x + 1)J_{n+2}^2 - x^2 J_{n+1}^2;$$

$$D = (x^2 + x)J_{n+2}^2 + x^3 J_n^2 + 2x^2 J_{n+2}J_n$$

$$= (x^2 + x)(J_{n+2} + x J_n)^2 - 2x^3 J_{n+2}J_n - x^4 J_n^2$$

$$= x(x + 1)J_{n+1}^2 - 2x^3 J_{n+2}J_n - x^4 J_n^2.$$
Thus

\[ S_1 + x^3 S_2 = (x + 1)j_{n+2}^2 + x(x + 1)j_{n+1}^2, \]

as expected. \( \square \)

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