Alternating Sums of the Reciprocal Fibonacci Numbers

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Abstract

In this paper, we investigate the alternating sums of the reciprocal Fibonacci numbers
\[ \sum_{k=n}^{m} (-1)^k / F_{ak+b}, \]
where \( a \in \{1, 2, 3\} \) and \( b < a \). The integer parts of the reciprocals of these sums are expressed explicitly in terms of the Fibonacci numbers.

1 Introduction

For an integer \( n \geq 0 \), the Fibonacci number \( F_n \) is defined recurrently by \( F_n = F_{n-1} + F_{n-2} \) with \( F_0 = 0 \) and \( F_1 = 1 \).

Recently, Ohtsuka and Nakamura [1] studied the infinite sums of the reciprocal Fibonacci numbers, and established the following result, where \( \lfloor \cdot \rfloor \) denotes the floor function.
Theorem 1. For all $n \geq 2$,
\[
\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n-2}, & \text{if } n \text{ is even;} \\
F_{n-2} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]


Theorem 2. If $m \geq 3$ and $n \geq 2$, then
\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n-2}, & \text{if } n \text{ is even;} \\
F_{n-2} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

In this article, we focus on the alternating sums of the reciprocal Fibonacci numbers
\[
\sum_{k=n}^{mn} (-1)^k \frac{1}{F_{ak+b}},
\]
where $a \in \{1, 2, 3\}$ and $b < a$. By evaluating the integer parts of these sums, we obtain several interesting families of identities concerning the Fibonacci numbers.

2 Results for $a = 1$

We first introduce several well-known results, which will be used throughout the article. The detailed proofs can be found in, for example, [3, Thm. 7, p. 9] and [2].

Lemma 3. For any positive integers $m$ and $n$, we have
\[
F_mF_n + F_{m+1}F_{n+1} = F_{m+n+1}.
\]

Lemma 4. For all $n \geq 1$, we have
\[
F_{2n+1} = F_{n+1}F_{n+2} - F_{n-1}F_n.
\]

Lemma 5. Let $a, b, c, d$ be positive integers with $a + b = c + d$ and $b \geq \max\{c, d\}$. Then
\[
F_aF_b - F_cF_d = (-1)^{a+1}F_{b-c}F_{b-d}.
\]

For the sake of argument, we present four auxiliary functions
\[
\begin{align*}
    f_1(n) &= \frac{1}{F_{n+1}} - \frac{(-1)^n}{F_n} - \frac{1}{F_{n+2}}, \\
    f_2(n) &= \frac{1}{F_{n+1} - 1} - \frac{(-1)^n}{F_n} - \frac{1}{F_{n+2} - 1}, \\
    f_3(n) &= \frac{-1}{F_{n+1} + 1} - \frac{(-1)^n}{F_n} + \frac{1}{F_{n+2} + 1}, \\
    f_4(n) &= \frac{-1}{F_{n+1} - 1} - \frac{(-1)^n}{F_n} + \frac{1}{F_{n+2}}.
\end{align*}
\]
It is clear that \( f_i(n) \) \((1 \leq i \leq 4)\) is positive if \( n \) is odd, and negative otherwise.

**Lemma 6.** If \( n \geq 2 \) is even, then

\[
f_1(n) + f_1(n + 1) < 0.
\]

**Proof.** Since \( n \) is even, it is straightforward to see

\[
f_1(n) + f_1(n + 1) &= \frac{2}{F_{n+1}} - \frac{1}{F_n} - \frac{1}{F_{n+3}} \\
&= \frac{(2F_n - F_{n+1})F_{n+3} - F_nF_{n+1}}{F_nF_{n+1}F_{n+3}} \\
&= \frac{F_{n-2}F_{n+3} - F_nF_{n+1}}{F_nF_{n+1}F_{n+3}} \\
&= \frac{-2}{F_nF_{n+1}F_{n+3}} \\
&< 0,
\]

where the last equality follows from Lemma 5 and the fact that \( n \) is even. \(\square\)

**Lemma 7.** For all \( n \geq 2 \), we have

\[
f_2(n) + f_2(n + 1) > 0.
\]

**Proof.** The statement is clearly true if \( n \) is odd. Thus, we focus on the case where \( n \) is even.

It follows from the definition of \( f_2(n) \) and Lemma 5 that

\[
f_2(n) + f_2(n + 1) &= \left( \frac{1}{F_{n+1} - 1} - \frac{1}{F_{n+3} - 1} \right) - \left( \frac{1}{F_n} - \frac{1}{F_{n+1}} \right) \\
&= \frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{F_{n-1}}{F_nF_{n+1}} \\
&= \frac{F_{n+1}(F_nF_{n+2} - F_{n-1}F_{n+3}) + F_{n-1}(F_{n+1} + F_{n+3} - 1)}{F_nF_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&= \frac{-2F_n + F_{n-1}(2F_{n+1} + F_{n+2} - 1)}{F_nF_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&= \frac{2(F_{n-1} - 1)F_{n+1} + F_{n-1}(F_{n+2} - 1)}{F_nF_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&> 0,
\]

which completes the proof. \(\square\)
Lemma 8. For all \( n \geq 2 \), we have
\[
\frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1} - 1} \geq 0.
\]

Proof. Applying Lemma 3, it is easy to see that, for \( n \geq 2 \),
\[
F_{2n+1} - 1 - 2F_n F_{n+1} = F_n^2 + F_{n+1}^2 - 2F_n F_{n+1} - 1 = (F_{n+1} - F_n)^2 - 1 \geq 0,
\]
from which we derive the conclusion that
\[
\frac{1}{F_{2n+1} - 1} \leq \frac{1}{2F_n F_{n+1}}.
\]

Therefore, we have
\[
\frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1} - 1} \geq \frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{2F_n F_{n+1}}
\]
\[
= \frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{2F_n F_{n+1}}{2F_n F_{n+1}},
\]
whose numerator is
\[
\psi(n) := 2F_n F_{n+1} F_{n+2} - (2F_n - 1)(F_{n+1} - 1)(F_{n+3} - 1).
\]

Applying Lemma 5 repeatedly and the fact \( F_{n+3} = 3F_{n+1} - F_{n-1} \), we can obtain
\[
\psi(n) = 2F_{n+1} (F_n F_{n+2} - F_{n-1} F_{n+3}) + 2F_{n-1} F_{n+1} + 2F_{n-1} F_{n+3} - F_{n+1} F_{n+3}
+ (F_{n+1} + F_{n+3}) - 2F_{n-1} - 1
= ((-1)^{n+1} + 1) 4F_{n+1} + 2F_{n-1} F_{n+1} + (2F_{n-1} - F_{n+1}) F_{n+3} - 3F_{n-1} - 1
= ((-1)^{n+1} + 1) 4F_{n+1} + F_{n-1} (2F_{n+1} - F_{n+2}) + (F_{n-1} F_{n+2} - F_{n-2} F_{n+3})
- 3F_{n-1} - 1
= ((-1)^{n+1} + 1) 4F_{n+1} + F_{n-1}^2 - 3F_{n-1} - 1 + (-1)^n 3.
\]

If \( n \) is even, we have \( \psi(n) = (F_{n-1} - 1)(F_{n-1} - 2) \geq 0 \). If \( n \) is odd, we have
\[
\psi(n) = (F_{n-1} + 1)(F_{n-1} + 4) + 8(F_n - 1) > 0.
\]

Therefore, \( \psi(n) \geq 0 \) always holds. This completes the proof.

Lemma 9. If \( n \geq 2 \) and \( m \geq 2 \), then
\[
f_2(n) + f_2(n + 1) + f_2(mn) + \frac{1}{F_{mn+2} - 1} > 0.
\]
Proof. If $mn$ is odd, then the result follows from Lemma 7 and the fact $f_2(mn) > 0$. So we assume that $mn$ is even. Now we have
\[ f_2(mn) + \frac{1}{F_{mn+2} - 1} = \frac{1}{F_{mn+1} - 1} - \frac{1}{F_{mn}} = \frac{-(F_{mn-1} - 1)}{F_{mn}(F_{mn+1} - 1)} > \frac{-1}{F_{mn+1} - 1}. \]
From the proof of Lemma 7 we know that whether $n$ is even or odd,
\[ f_2(n) + f_2(n + 1) \geq \frac{F_{n+2}}{(F_n+1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}}. \]
Therefore,
\[ f_2(n) + f_2(n + 1) + f_2(mn) + \frac{1}{F_{mn+2} - 1} > \frac{F_{n+2}}{(F_n+1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{mn+1} - 1} \]
\[ \geq \frac{F_{n+2}}{(F_n+1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1} - 1} \]
\[ \geq 0, \]
where the last inequality follows from Lemma 8. □

Employing the fact $2(F_{2n+2} + 1) \geq (F_n+1)(F_{n+3} + 1)$ and similar arguments in the proof of Lemma 8, we have the following result, whose proof is omitted here.

Lemma 10. If $n \geq 5$ is odd, then
\[ f_3(n) + f_3(n + 1) > \frac{1}{F_{2n+2} + 1}. \]

Now we establish two properties about $f_4(n)$.

Lemma 11. For $n \geq 1$, we have
\[ f_4(n) + f_4(n + 1) < 0. \]

Proof. If $n$ is even, the result follows from the definition of $f_4(n)$. Next we consider the case where $n$ is odd. Applying the argument in the proof of Lemma 6, we can easily deduce that
\[ f_4(n) + f_4(n + 1) = \frac{-2}{F_{n+1}} + \frac{1}{F_n} + \frac{1}{F_{n+3}} = \frac{-2}{F_n F_{n+1} F_{n+3}} < 0. \]
This completes the proof. □

Lemma 12. If $n \geq 1$ and $m \geq 2$, then
\[ f_4(n) + f_4(n + 1) + f_4(mn) < 0. \]
Proof. If \( mn \) is even, the result follows from Lemma 11 and the fact \( f_4(mn) < 0 \). So we assume that \( mn \) is odd, which implies that \( m \geq 3 \) and \( n \) is odd. Since \( mn \) is odd, we have

\[
f_4(mn) = \frac{-1}{F_{mn+1}} + \frac{1}{F_{mn}} + \frac{1}{F_{mn+2}} < \frac{1}{F_{mn}} \leq \frac{1}{F_{3n}}.
\]

Now we have

\[
f_4(n) + f_4(n + 1) + f_4(mn) < \frac{-2}{F_nF_{n+1}F_{n+3}} + \frac{1}{F_{3n}}.
\]

To complete the proof, we only need to show that \( 2F_{3n} > F_nF_{n+1}F_{n+3} \).

It follows from Lemma 3 that \( F_{2n+2} = F_{n-1}F_{n+2} + F_nF_{n+3} \), which implies \( F_nF_{n+1}F_{n+3} < F_{n+1}F_{2n+2} \). Furthermore, employing Lemma 3 again, we can conclude that

\[
F_{n+1}F_{2n+2} = (F_{n-1} + F_n)(F_{2n} + F_{2n+1})
\]
\[
= (F_{n-1}F_{2n} + F_nF_{2n+1}) + F_{n-1}F_{2n+1} + F_nF_{2n}
\]
\[
= F_{3n} + F_{n-1}F_{2n+1} + F_{n+1}F_{2n} - F_{n-1}F_{2n}
\]
\[
= F_{3n} + (F_{n-1}F_{2n-1} + F_{n+1}F_{2n})
\]
\[
< 2F_{3n},
\]

which completes the proof.

\[\Box\]

**Theorem 13.** If \( n \geq 4 \) and \( m \geq 2 \), then

\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} \right)^{-1} \right\rfloor = \begin{cases} 
F_{n+1} - 1, & \text{if } n \text{ is even;} \\
-F_{n+1} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

Proof. We first consider the case where \( n \) is even. It follows from Lemma 6 that

\[
\sum_{k=n}^{mn-1} f_1(k) < 0.
\]

It is clear that \( mn \) is even, which ensures that

\[
f_1(mn) + \frac{1}{F_{mn+2}} < 0.
\]

With the help of \( f_1(n) \) and the above two inequalities, we can obtain

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{1}{F_{n+1}} - \left( \frac{1}{F_{mn+2} + f_1(mn)} \right) - \sum_{k=n}^{mn-1} f_1(k) > \frac{1}{F_{n+1}}.
\]
Applying Lemma 7 and Lemma 9, we have

\[ \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{1}{F_{n+1}} - \left( f_2(n) + f_2(n+1) + f_2(mn) + \frac{1}{F_{mn+2} - 1} \right) - \sum_{k=n+2}^{mn-1} f_2(k) < \frac{1}{F_{n+1} - 1}. \]

Therefore, we obtain

\[ \frac{1}{F_{n+1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} < \frac{1}{F_{n+1} - 1}, \]

which shows that the statement is true when \( n \) is even.

We now turn to consider the case where \( n \geq 5 \) is odd. If \( mn \) is odd, it is easy to see that

\[ f_3(mn) - \frac{1}{F_{mn+2} + 1} > 0. \]

Lemma 10 tells us that \( f_3(n) + f_3(n+1) > 0 \). Therefore,

\[ \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{-1}{F_{n+1} + 1} - \sum_{k=n}^{mn-1} f_3(k) - \left( f_3(mn) - \frac{1}{F_{mn+2} + 1} \right) < \frac{-1}{F_{n+1} + 1}. \]

If \( mn \) is even, employing Lemma 10 again, we can deduce

\[ \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{-1}{F_{n+1} + 1} - \sum_{k=n+2}^{mn} f_3(k) - \left( f_3(n) + f_3(n+1) - \frac{1}{F_{2n+2} + 1} \right) \leq \frac{-1}{F_{n+1} + 1} - \sum_{k=n+2}^{mn} f_3(k) - \left( f_3(n) + f_3(n+1) - \frac{1}{F_{2n+2} + 1} \right) < \frac{-1}{F_{n+1} + 1}. \]

Now we can conclude that if \( n \geq 5 \) is odd, then

\[ \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} < \frac{-1}{F_{n+1} + 1}. \]

If \( mn \) is even, then Lemma 11 implies that

\[ \sum_{k=n}^{mn} f_4(k) < 0. \]
If \( mn \) is odd, invoking Lemma 11 and Lemma 12, we can get

\[
\sum_{k=n}^{mn} f_4(k) = \sum_{k=n}^{mn-1} f_4(k) + (f_4(n) + f_4(n + 1) + f_4(mn)) < 0.
\]

Thus, we always have

\[
\sum_{k=n}^{mn} f_4(k) < 0,
\]

from which we obtain

\[
\sum_{k=n}^{mn} (-1)^k \frac{1}{F_k} = -\frac{1}{F_{n+1}} + \frac{1}{F_{mn+2}} - \sum_{k=n}^{mn} f_4(k) > -\frac{1}{F_{n+1}}.
\]

Therefore, we arrive at

\[
-\frac{1}{F_{n+1}} < \sum_{k=n}^{mn} (-1)^k \frac{1}{F_k} < -\frac{1}{F_{n+1} + 1},
\]

which shows that the result holds for odd \( n \).

\[\] 3 Results for \( a = 2 \)

We first introduce the following notations

\[
\begin{align*}
g_1(n) &= \frac{1}{F_{2n-2} + F_{2n}} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n} + F_{2n+2}}, \\
g_2(n) &= \frac{1}{F_{2n-2} + F_{2n} - 1} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n} + F_{2n+2} - 1}, \\
g_3(n) &= \frac{1}{F_{2n-2} + F_{2n} + 1} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n} + F_{2n+2} + 1}, \\
g_4(n) &= \frac{-1}{F_{2n-2} + F_{2n}} - \frac{(-1)^n}{F_{2n}} + \frac{1}{F_{2n} + F_{2n+2}}, \\
g_5(n) &= \frac{-1}{F_{2n-2} + F_{2n} + 1} - \frac{(-1)^n}{F_{2n}} + \frac{1}{F_{2n} + F_{2n+2} + 1}.
\end{align*}
\]

It is routine to check that for \( 1 \leq i \leq 5 \), \( g_i(n) \) is positive if \( n \) is odd, and negative otherwise.

**Lemma 14.** If \( n \geq 1 \), then \( g_1(n) + g_1(n + 1) > 0 \) and

\[
g_1(n) + g_1(n + 1) > g_1(n + 2) + g_1(n + 3).
\]
Proof. If \( n \) is odd, we have
\[
g_1(n) + g_1(n + 1) = \left( \frac{1}{F_{2n-2} + F_{2n}} - \frac{1}{F_{2n+2} + F_{2n+4}} \right) + \left( \frac{1}{F_{2n}} - \frac{1}{F_{2n+2}} \right)
\]
\[
= \frac{5F_{2n+1}}{(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} + \frac{F_{2n+1}}{F_{2n}F_{2n+2}}
\]
\[
> 0.
\]
Applying the easily checked fact
\[
\frac{F_{2n+1}}{F_{2n-2} + F_{2n}} > \frac{F_{2n+5}}{F_{2n+6} + F_{2n+8}},
\]
\[
\frac{F_{2n+1}}{F_{2n}F_{2n+2}} > \frac{F_{2n+5}}{F_{2n+4}F_{2n+6}},
\]
we can conclude that \( g_1(n) + g_1(n + 1) > g_1(n + 2) + g_1(n + 3) \).

Now we consider the case where \( n \) is even. Doing some elementary manipulations and using Lemma 5, we have
\[
g_1(n) + g_1(n + 1) = \left( \frac{1}{F_{2n-2} + F_{2n}} - \frac{1}{F_{2n+2} + F_{2n+4}} \right) + \left( \frac{1}{F_{2n+2}} - \frac{1}{F_{2n+2} + F_{2n+4}} \right)
\]
\[
= \frac{F_{2n-2}(F_{2n}F_{2n+4} - F_{2n+2}^2) + (F_{2n}^2 - F_{2n-2}F_{2n+2})F_{2n+4}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}
\]
\[
= \frac{F_{2n+4} - F_{2n-2}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}
\]
\[
= \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}
\]
\[
> 0.
\]
Applying the above identity, we see that
\[
\frac{g_1(n) + g_1(n + 1)}{g_1(n + 2) + g_1(n + 3)} = \frac{F_{2n+1}F_{2n+4}F_{2n+6}}{F_{2n}F_{2n+2}F_{2n+5}} \frac{F_{2n+6} + F_{2n+8}}{F_{2n-2} + F_{2n}} > 1.
\]
Thus, \( g_1(n) + g_1(n + 1) > g_1(n + 2) + g_1(n + 3) \) also holds. \( \square \)

**Lemma 15.** For \( n \geq 1 \), we have
\[
F_{6n+2} > F_{2n}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}).
\]

**Proof.** It follows from Lemma 5 that
\[
F_{2n-1}F_{2n+3} - F_{2n-2}F_{2n+4} = 5,
\]
\[
F_{2n-1}F_{2n+1} - F_{2n}^2 = 1,
\]
\[
F_{2n+1}F_{2n+3} - F_{2n}F_{2n+4} = 2.
\]
Thus, \( F_{2n-1}F_{2n+3} > F_{2n-2}F_{2n+4}, F_{2n-1}F_{2n+1} > F_{2n}^2, \) and \( F_{2n+1}F_{2n+3} > F_{2n}F_{2n+4}. \)

Employing Lemma 3 repeatedly and the above three inequalities, we have

\[
F_{6n+2} = F_{2n}F_{4n+1} + F_{2n+1}F_{4n+2} \\
= F_{2n}(F_{2n-2}F_{2n+2} + F_{2n-1}F_{2n+3}) + F_{2n+1}(F_{2n-1}F_{2n+2} + F_{2n}F_{2n+3}) \\
> F_{2n-2}F_{2n}F_{2n+2} + F_{2n-2}F_{2n+4}F_{2n} + F_{2n}^2F_{2n+2} + F_{2n}F_{2n+1}F_{2n} \\
= F_{2n}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}),
\]

which completes the proof. \( \square \)

**Lemma 16.** If \( n \geq 1 \) and \( m \geq 3 \), then

\[
g_1(n) + g_1(n + 1) + g_1(mn) > 0.
\]

**Proof.** If \( mn \) is odd, then the result follows from Lemma 14 and the fact \( g_1(mn) > 0 \). Thus we focus on the case where \( mn \) is even. For \( k \geq 1 \),

\[
\frac{1}{F_{2k-2} + F_{2k}} - \frac{1}{F_{2k}} = -\frac{F_{2k-2}}{(F_{2k-2} + F_{2k})F_{2k}} \\
= -\frac{F_{2k-2}}{F_{2k-2}F_{2k} + F_{2k}^2} \\
> -\frac{F_{2k-2}}{F_{2k-2}F_{2k+2}} \\
= -\frac{1}{F_{2k+2}},
\]

where the inequality follows from \( F_{2k}^2 - F_{2k-2}F_{2k+2} = 1 \). Since \( mn \) is even, employing the above inequality, we have

\[
g_1(mn) > -\frac{1}{F_{2mn+2}} - \frac{1}{F_{2mn} + F_{2mn+2}} > -\frac{2}{F_{2mn+2}} \geq -\frac{2}{F_{6n+2}}.
\]

From the proof of Lemma 14 we know that whether \( n \) is even or odd, we always have

\[
g_1(n) + g_1(n + 1) \geq \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})},
\]

Therefore,

\[
g_1(n) + g_1(n + 1) + g_1(mn) > \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{2}{F_{6n+2}} \\
> \frac{2}{F_{2n}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{2}{F_{6n+2}} \\
> 0,
\]

where the last inequality follows from Lemma 15. \( \square \)
Lemma 17. If \( n > 0 \), then
\[
2F_{4n}(F_{4n} + F_{4n+2}) > F_{2n+2}F_{4n+3}(F_{2n-2} + F_{2n}).
\]

Proof. It suffices to show that \( 2F_{4n}^2 > F_{2n-2}F_{2n+2}F_{4n+3} \) and \( 2F_{4n}F_{4n+2} > F_{2n}F_{2n+2}F_{4n+3} \). These two inequalities can be proved using similar arguments, so we only prove the first one.

Applying Lemma 5 repeatedly and Lemma 3, we can obtain
\[
2F_{4n}^2 = 2F_{4n-3}F_{4n+3} - 8
\]
\[
= 2(F_{2n-2}^2 + F_{2n-1}^2)F_{4n+3} - 8
\]
\[
> (F_{2n-2}F_{2n-1} + 2F_{2n-1}^2)F_{4n+3} - 8
\]
\[
= F_{2n-1}F_{2n+1}F_{4n+3} - 8
\]
\[
= (F_{2n-2}F_{2n+2} + 2)F_{4n+3} - 8
\]
\[
> F_{2n-2}F_{2n+2}F_{4n+3}.
\]

The proof is completed. \( \square \)

Lemma 18. For all \( n \geq 2 \), we have
\[
g_2(n) + g_2(n + 1) + g_2(2n) > 0.
\]

Proof. It is straightforward to verify that \( F_{2n-2} + F_{2n} + F_{2n+2} + F_{2n+4} = 3(F_{2n} + F_{2n+2}) \). Applying Lemma 5 repeatedly, we get
\[
(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) = F_{2n-2}F_{2n+2} + F_{2n-2}F_{2n+4} + F_{2n}F_{2n+2} + F_{2n}F_{2n+4}
\]
\[
= F_{2n-2}F_{2n+2} + (F_{2n}F_{2n+2} - 3) + F_{2n}F_{2n+2}
\]
\[
+ F_{2n}(2F_{2n+2} + F_{2n+1})
\]
\[
= (F_{2n-2}F_{2n+2} + F_{2n}^2) + (F_{2n}^2 + F_{2n}F_{2n+1})
\]
\[
+ 4F_{2n}F_{2n+2} - 3
\]
\[
= 5F_{2n}F_{2n+2} - 4.
\]

It follows from the definition of \( g_2(n) \) and the above two equations that
\[
g_2(n) + g_2(n + 1) \geq \left( \frac{1}{F_{2n-2} + F_{2n} - 1} - \frac{1}{F_{2n+2} + F_{2n+4} - 1} \right) - \left( \frac{1}{F_{2n}} - \frac{1}{F_{2n+2}} \right)
\]
\[
= \frac{5F_{2n+1}}{(F_{2n-2} + F_{2n} - 1)(F_{2n+2} + F_{2n+4} - 1)} - \frac{F_{2n+1}}{F_{2n}F_{2n+2}}
\]
\[
= \frac{3(F_{2n} + F_{2n+2} + 1)F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n} - 1)(F_{2n+2} + F_{2n+4} - 1)}
\]
\[
> \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n} - 1)},
\]
where the last inequality follows from \(3F_n > F_{n+2}\).

It is routine to show
\[
2(F_{4n+2} - F_{4n-2}) = 2(2F_{4n} + F_{4n-1} - F_{4n-2}) \\
= 3F_{4n} + F_{4n} + 2F_{4n-3} \\
> 3F_{4n} + (2F_{4n-2} + F_{4n-3}) + F_{4n-2} \\
> 3(F_{4n-2} + F_{4n}),
\]
which means
\[
F_{4n+2} - F_{4n-2} > \frac{3}{2}(F_{4n-2} + F_{4n}).
\]

Employing the above inequality, we can deduce that
\[
g_2(2n) = \frac{F_{4n+2} - F_{4n-2}}{(F_{4n-2} + F_{4n} - 1)(F_{4n} + F_{4n+2} - 1)} - \frac{1}{F_{4n}} \\
> \frac{3}{2(F_{4n} + F_{4n+2} - 1)} - \frac{1}{F_{4n}} \\
= -\frac{F_{4n+3} + 2}{2F_{4n}(F_{4n} + F_{4n+2} - 1)} \\
> -\frac{F_{4n+3}^2}{2F_{4n}(F_{4n} + F_{4n+2} - 1)}.
\]

Now we conclude that
\[
g_2(n) + g_2(n + 1) + g_2(2n) > \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n-1} - 1)} - \frac{F_{4n+3}}{2F_{4n}(F_{4n} + F_{4n+2} - 1)} > 0,
\]
where the last inequality follows from Lemma 17.

Applying the argument in the proof of Lemma 18, it can be readily seen the following property of \(g_3(n)\), whose proof is omitted here.

**Lemma 19.** If \(n \geq 2\) is even, we have
\[
g_3(n) + g_3(n + 1) < 0.
\]

Imitating the proof of Lemma 14 and Lemma 16 respectively, we can easily get the following results on \(g_4(n)\).

**Lemma 20.** For \(n \geq 1\), we have
\[
g_4(n) + g_4(n + 1) < 0.
\]
Lemma 21. If \( n \geq 1 \) and \( m \geq 2 \), then
\[
g_4(n) + g_4(n + 1) + g_4(mn) < 0.
\]

Lemma 22. If \( n \geq 1 \) is odd, we have
\[
g_5(n) + g_5(n + 1) > \frac{1}{F_{4n} + F_{4n+2} + 1}.
\]

Proof. It is easy to see that the result is true for \( n = 1 \), thus we assume that \( n \geq 3 \). From the proof of Lemma 18, we can easily obtain that if \( n \geq 3 \) is odd, then
\[
g_5(n) + g_5(n + 1) = \frac{3(F_{2n} + F_{2n+2} - 1)F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n} + 1)(F_{2n+2} + F_{2n+4} + 1)} > \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n} + 1)}.
\]

Employing Lemma 3 repeatedly, it is easy to see that
\[
F_{2n+2}(F_{2n-2} + F_{2n} + 1) < F_{2n-2}F_{2n+3} + F_{2n}F_{2n+3} + F_{2n+2} = F_{4n} - F_{2n-3}F_{2n+2} + F_{4n+2} - F_{2n-1}F_{2n+2} + F_{2n+2} < F_{4n} + F_{4n+2}.
\]

Combining the above two inequalities yields the desired result. \( \square \)

Lemma 23. For \( n \geq 2 \), we have
\[
F_{4n-2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) > F_{4n}(F_{4n-2} + F_{4n}).
\]

Proof. We first consider the right-hand side. Applying \( F_{4n}^2 - F_{4n-1}F_{4n+1} = -1 \), we have
\[
F_{4n}(F_{4n-2} + F_{4n}) = F_{4n-2}F_{4n} + F_{4n}^2 = F_{4n-2}F_{4n} + F_{4n-1}F_{4n+1} - 1 = F_{8n-1} - 1.
\]

For the left-hand side, we have that if \( n \geq 2 \), then
\[
(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) = F_{2n-2}F_{2n+2} + F_{2n}F_{2n+2} + F_{2n-2}F_{2n+4} + F_{2n}F_{2n+4} > (F_{2n-2}F_{2n+1} + F_{2n-1}F_{2n+2}) + (F_{2n-2}F_{2n+3} + F_{2n-1}F_{2n+4}) + F_{2n-2}F_{2n+4} > F_{4n} + F_{4n+2} + 2.
\]

Therefore, using the fact \( F_{4n-2}F_{4n+2} - F_{4n-1}F_{4n+1} = -2 \), we have
\[
F_{4n-2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) > F_{4n-2}F_{4n} + F_{4n-2}F_{4n+2} + 2 = F_{4n-2}F_{4n} + F_{4n-1}F_{4n+1} = F_{8n-1}.
\]

Thus the left-hand side is greater than the right-hand side. \( \square \)
Theorem 24. If \( n \geq 2 \) is even and \( m \geq 2 \), then
\[
\left| \left( \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} \right)^{-1} \right| = \begin{cases} 
F_{2n-2} + F_{2n} - 1, & \text{if } m = 2; \\
F_{2n-2} + F_{2n}, & \text{if } m > 2.
\end{cases}
\]

Proof. We first consider the case where \( m = 2 \). From Lemma 14 we know that
\[
\sum_{k=n}^{2n-1} g_1(k) < \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} \cdot \frac{n}{2} < \frac{1}{(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}.
\]
In addition,
\[
g_1(2n) + \frac{1}{F_{4n} + F_{4n+2}} = \frac{1}{F_{4n-2} + F_{4n}} - \frac{1}{F_{4n}} = \frac{-F_{4n-2}}{F_{4n}(F_{4n-2} + F_{4n})}.
\]
Therefore, invoking Lemma 23, we have
\[
\sum_{k=n}^{2n} g_1(k) + \frac{1}{F_{4n} + F_{4n+2}} < \frac{1}{(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{F_{4n-2}}{F_{4n}(F_{4n-2} + F_{4n})} < 0.
\]
Now with the help of \( g_1(n) \), we can obtain
\[
\sum_{k=n}^{2n} (-1)^k \frac{1}{F_{2k}} = \frac{1}{F_{2n-2} + F_{2n}} - \frac{1}{F_{4n} + F_{4n+2}} - \sum_{k=n}^{2n} g_1(k) > \frac{1}{F_{2n-2} + F_{2n}}.
\]
From the proof of Lemma 18, we know that \( g_2(n) + g_2(n + 1) > 0 \). Moreover, applying Lemma 18, we can deduce
\[
\sum_{k=n}^{2n} g_2(k) = g_2(n) + g_2(n + 1) + g_2(2n) + \sum_{k=n+2}^{2n-1} g_2(k) > 0.
\]
Therefore,
\[
\sum_{k=n}^{2n} (-1)^k \frac{1}{F_{2k}} = \frac{1}{F_{2n-2} + F_{2n} - 1} - \frac{1}{F_{4n} + F_{4n+2} - 1} - \sum_{k=n}^{2n} g_2(k) < \frac{1}{F_{2n-2} + F_{2n} - 1}.
\]
We now conclude that
\[
\frac{1}{F_{2n-2} + F_{2n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2} + F_{2n} - 1},
\]
which shows that the statement for \( m = 2 \) is true.
Next we turn to consider the case where \( m > 2 \). First, employing Lemma 14 and Lemma 16, we see that

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2} + F_{2n}} - (g_1(n) + g_1(n+1) + g_1(mn)) - \sum_{k=n+2}^{mn-1} g_1(k) \times \frac{1}{F_{2n-2} + F_{2n}}.
\]

We write the sum in terms of \( g_3(n) \) as

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} = \frac{1}{F_{2n-2} + F_{2n} + 1} - \sum_{k=n}^{mn-1} g_3(k) - \left( g_3(mn) + \frac{1}{F_{2mn} + F_{2mn+2} + 1} \right)
\]

\[
> \frac{1}{F_{2n-2} + F_{2n} + 1},
\]

where the last inequality follows from Lemma 19. Now we get

\[
\frac{1}{F_{2n-2} + F_{2n} + 1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2} + F_{2n}},
\]

which yields the desired identity. \( \square \)

**Theorem 25.** If \( n \geq 1 \) is odd and \( m \geq 2 \), then

\[
\left\lfloor \left( \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} \right)^{-1} \right\rfloor = -F_{2n-2} - F_{2n} - 1.
\]

**Proof.** If \( mn \) is even, it follows from Lemma 20 that

\[
\sum_{k=n}^{mn} g_4(k) < 0.
\]

If \( mn \) is odd, then Lemma 20 and Lemma 21 ensure that

\[
\sum_{k=n}^{mn} g_4(k) = \sum_{k=n+2}^{mn-1} g_4(k) + (g_4(n) + g_4(n+1) + g_4(mn)) < 0.
\]

Therefore, we always have

\[
\sum_{k=n}^{mn} g_4(k) < 0.
\]

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With the help of $g_4(n)$, we have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} = \frac{-1}{F_{2n-2} + F_{2n}} + \frac{1}{F_{2mn-2} + F_{2mn}} - \sum_{k=n}^{mn} g_4(k) > \frac{-1}{F_{2n-2} + F_{2n}}.$$

From Lemma 22 we know that if $n$ is odd, then $g_5(n) + g_5(n+1) > 0$. Now we claim that

$$\sum_{k=n}^{mn} g_5(k) > \frac{1}{F_{2mn} + F_{2mn+2} + 1}.$$

If $mn$ is even, employing Lemma 22, we obtain

$$\sum_{k=n}^{mn} g_5(k) - \frac{1}{F_{2mn} + F_{2mn+2} + 1} \geq \sum_{k=n}^{mn} g_5(k) - \frac{1}{F_{4n} + F_{4n+2} + 1} \geq g_5(n) + g_5(n+1) - \frac{1}{F_{4n} + F_{4n+2} + 1} > 0.$$

If $mn$ is odd, then

$$\sum_{k=n}^{mn} g_5(k) - \frac{1}{F_{2mn} + F_{2mn+2} + 1} = \sum_{k=n}^{mn-1} g_5(k) + \left(g_5(mn) - \frac{1}{F_{2mn} + F_{2mn+2} + 1}\right) > -\frac{1}{F_{2mn} + F_{2mn+2} + 1} + \frac{1}{F_{2mn}} > 0.$$

Therefore, we have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} = \frac{-1}{F_{2n-2} + F_{2n} + 1} + \frac{1}{F_{2mn-2} + F_{2mn} + 1} - \sum_{k=n}^{mn} g_5(k) < \frac{-1}{F_{2n-2} + F_{2n} + 1}.$$

Now we can conclude that

$$\frac{-1}{F_{2n-2} + F_{2n}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} < \frac{-1}{F_{2n-2} + F_{2n} + 1},$$

from which the desired result follows.

Similarly, we can prove the following results.

**Theorem 26.** If $n \geq 4$ is even and $m \geq 2$, then

$$\left[\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}}\right]^{-1} = F_{2n-3} + F_{2n-1} - 1.$$
Theorem 27. If \( n \geq 3 \) is odd and \( m \geq 2 \), then

\[
\left[ \sum_{k=n}^{mn} (-1)^k \frac{F_{2k-1}}{F_{3n}} \right]^{-1} = \begin{cases} 
-F_{2n-3} - F_{2n-1} - 1, & \text{if } m = 2; \\
-F_{2n-3} - F_{2n-1}, & \text{if } m > 2.
\end{cases}
\]

4 Results for \( a = 3 \)

We first introduce the following notations:

\[
\begin{align*}
s_1(n) &= \frac{1}{2F_{3n-1}} - \frac{(-1)^n}{F_{3n}} - \frac{1}{2F_{3n+2}}, \\
s_2(n) &= \frac{1}{2F_{3n-1} - 1} - \frac{(-1)^n}{F_{3n}} - \frac{1}{2F_{3n+2} - 1}, \\
s_3(n) &= \frac{-1}{2F_{3n-1} - 1} - \frac{(-1)^n}{F_{3n}} + \frac{1}{2F_{3n+2}}, \\
s_4(n) &= \frac{-1}{2F_{3n-1} + 1} - \frac{(-1)^n}{F_{3n}} + \frac{1}{2F_{3n+2} + 1}.
\end{align*}
\]

It is easy to see that for each \( i \), \( s_i(n) \) is positive if \( n \) is odd, and negative otherwise.

Lemma 28. If \( n \geq 2 \) is even, then

\[ s_1(n) + s_1(n + 1) < 0. \]

Proof. Since \( n \) is even, applying Lemma 5 twice, we have

\[
\begin{align*}
s_1(n) + s_1(n + 1) &= \left( \frac{1}{2F_{3n-1} - 1} - \frac{1}{F_{3n}} - \frac{1}{2F_{3n+2}} - \frac{1}{F_{3n}} - \frac{1}{2F_{3n+2}} \right) \\
&= \frac{2F_{3n+2}}{F_{3n-1}F_{3n+5} - 2F_{3n+1}F_{3n+3} - 2F_{3n+1}F_{3n+3}} \\
&= 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+3} - F_{3n-1}F_{3n+1}F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\
&= 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+3} - (F_{3n}^2 + 1)F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\
&= 2 \cdot \frac{F_{3n}(F_{3n+2}F_{3n+3} - F_{3n}F_{3n+5}) - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\
&= 2 \cdot \frac{2F_{3n} - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\
&< 0,
\end{align*}
\]

which completes the proof. \( \square \)
Lemma 29. For all $n \geq 1$, we have

$$s_2(n) + s_2(n + 1) > 0.$$ 

Proof. It is clear that the result holds if $n$ is odd. In the rest, we assume that $n$ is even. Applying the analysis in the proof of Lemma 28, we can easily obtain

$$s_2(n) + s_2(n + 1) = \left(\frac{1}{2F_{3n-1} - 1} - \frac{1}{2F_{3n+5} - 1}\right) - \left(\frac{1}{F_{3n}} - \frac{1}{F_{3n+3}}\right)$$

$$= \frac{8F_{3n+2}}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)} - \frac{2F_{3n+1}}{F_{3n}F_{3n+3}}$$

$$= \frac{8(F_{3n}F_{3n+2}F_{3n+3} - F_{3n-1}F_{3n+1}F_{3n+5}) + 2F_{3n+1}(2F_{3n-1} + 2F_{3n+5} - 1)}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)F_{3n}F_{3n+3}}$$

$$= \frac{16F_{3n} - 8F_{3n+5} + 2F_{3n+1}(2F_{3n-1} + 2F_{3n+5} - 1)}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)F_{3n}F_{3n+3}}$$

$$> \frac{4F_{3n+1}F_{3n+5} - 8F_{3n+5}}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)F_{3n}F_{3n+3}}$$

$$> 0.$$ 

The proof is completed. \qed

Lemma 30. If $n \geq 1$ and $m \geq 2$, then

$$s_2(n) + s_2(n + 1) + s_2(mn) > 0.$$ 

Proof. If $mn$ is odd, then the result follows from Lemma 29 and the fact $s_2(mn) > 0$. So we assume that $mn$ is even. Now it is clear that

$$s_2(mn) = \frac{1}{2F_{3mn-1} - 1} - \frac{1}{F_{3mn}} - \frac{1}{2F_{3mn+2} - 1} > -\frac{1}{F_{3mn}} \geq -\frac{1}{F_{6n}}.$$ 

If $n$ is odd, we have

$$s_2(n) + s_2(n + 1) > \frac{1}{F_{3n}} - \frac{1}{F_{3n+3}} = \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} > \frac{2}{F_{3n}F_{3n+3}}.$$ 

If $n$ is even, then from Lemma 29 we know that

$$s_2(n) + s_2(n + 1) > \frac{4F_{3n+1}F_{3n+5} - 8F_{3n+5}}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)F_{3n}F_{3n+3}}$$

$$= \frac{4F_{3n+5}(2F_{3n-1} + F_{3n-2} - 2)}{(2F_{3n-1} - 1)(2F_{3n+5} - 1)F_{3n}F_{3n+3}}$$

$$> \frac{2}{F_{3n}F_{3n+3}}.$$ 

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Now we can derive the conclusion that
\[ s_2(n) + s_2(n+1) + s_2(mn) > \frac{2}{F_{3n}F_{3n+3}} - \frac{1}{F_{6n}} \geq 0, \]
where the last inequality follows from
\[ 2F_{6n} = F_{3n}(2F_{3n-1} + 2F_{3n+1}) > F_{3n}(F_{3n} + 2F_{3n+1}) = F_{3n}F_{3n+3}. \]
This completes the proof. \( \Box \)

**Lemma 31.** For all \( n \geq 1, \)
\[ s_3(n) + s_3(n+1) < 0. \]

**Proof.** The result clearly holds when \( n \) is even. If \( n \) is odd, applying similar analysis in the proof of Lemma 28, we can easily derive
\[ s_3(n) + s_3(n+1) = 2 \cdot \frac{2F_{3n} - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} < 0, \]
which completes the proof. \( \Box \)

**Lemma 32.** If \( n \geq 1 \) and \( m \geq 2, \) then
\[ s_3(n) + s_3(n+1) + s_3(mn) < 0. \]

**Proof.** If \( mn \) is even, then the result follows from Lemma 31 and the fact \( s_3(mn) < 0. \) Now we assume that \( mn \) is odd, which implies that \( n \) is odd and \( m \geq 3. \) First we have
\[ s_3(mn) = \frac{-1}{2F_{3mn-1}} + \frac{1}{F_{3mn}} + \frac{1}{2F_{3mn+2}} < \frac{1}{F_{3mn}} \leq \frac{1}{F_{9n}}. \]
Moreover, from the proof of Lemma 31 we know
\[ s_3(n) + s_3(n+1) = -\frac{2(F_{3n+5} - 2F_{3n})}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} < -\frac{1}{F_{3n-1}F_{3n}F_{3n+3}}. \]
Now we arrive at
\[ s_3(n) + s_3(n+1) + s_3(mn) < -\frac{1}{F_{3n-1}F_{3n}F_{3n+3}} + \frac{1}{F_{9n}} < 0, \]
where the last inequality follows from
\[ F_{9n} = F_{3n-2}F_{6n+1} + F_{3n-1}F_{6n+2} > F_{3n-1}(F_{3n-1}F_{3n+2} + F_{3n}F_{3n+3}) > F_{3n-1}F_{3n}F_{3n+3}. \]
The proof is completed. \( \Box \)
Lemma 33. If \( n \geq 1 \) is odd, then

\[ s_4(n) + s_4(n + 1) > \frac{1}{2F_{6n+2} + 1}. \]

Proof. It is easy to check that the result holds for \( n = 1 \), so we assume that \( n \geq 3 \). Applying the similar analysis in the proof of Lemma 28, we have that, for \( n \geq 3 \),

\[
s_4(n) + s_4(n + 1) = -\left( \frac{1}{2F_{3n-1} + 1} - \frac{1}{2F_{3n+5} + 1} \right) + \left( \frac{1}{F_{3n}} - \frac{1}{F_{3n+3}} \right)
\]

\[
= -\left( \frac{2F_{3n+5} - 2F_{3n-1}}{(2F_{3n-1} + 1)(2F_{3n+5} + 1)} \right) + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}}
\]

\[> -\frac{F_{3n+5} - F_{3n-1}}{(2F_{3n-1} + 1)F_{3n+5}} + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}}
\]

\[
= -\frac{4F_{3n+2}}{(2F_{3n-1} + 1)F_{3n+5}} + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}}
\]

\[
= \frac{4(F_{3n-1}F_{3n+1}F_{3n+5} - F_{3n}F_{3n+2}F_{3n+3}) + 2F_{3n+1}F_{3n+5}}{(2F_{3n-1} + 1)F_{3n}F_{3n+3}F_{3n+5}}
\]

\[
= \frac{4(2F_{3n} - F_{3n+5}) + 2F_{3n+1}F_{3n+5}}{(2F_{3n-1} + 1)F_{3n}F_{3n+3}F_{3n+5}}
\]

\[> \frac{(2F_{3n+1} - 4)F_{3n+5}}{(2F_{3n-1} + 1)F_{3n}F_{3n+3}F_{3n+5}}
\]

\[> \frac{1}{F_{3n}F_{3n+3}}.
\]

In addition, we have

\[ F_{2n+2} = F_{n-1}F_{n+2} + F_nF_{n+3} > F_nF_{n+3}. \]

Combining the above two inequalities together yields the desired result. \( \square \)

Theorem 34. If \( n \geq 1 \) and \( m \geq 2 \), then

\[
\left[ \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} \right]^{-1} = \begin{cases} 
2F_{3n-1} - 1, & \text{if } n \text{ is even;} \\
-2F_{3n-1} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]
Proof. We first consider the case where $n$ is even. With the help of $s_1(n)$, we have

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k) - \left( s_1(mn) + \frac{1}{2F_{3mn+2}} \right)
\]

\[
= \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k) - \left( \frac{1}{2F_{3mn+2}} - \frac{1}{F_{3mn}} \right)
\]

\[
> \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k)
\]

\[
> \frac{1}{2F_{3n-1}},
\]

where the last inequality follows from Lemma 28.

Employing Lemma 29 and Lemma 30, we can deduce that

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{1}{2F_{3n-1} - 1} - \frac{1}{2F_{3mn+2} - 1} - \sum_{k=n+2}^{mn-1} s_2(k) - (s_2(n) + s_2(n+1) + s_2(mn))
\]

\[
< \frac{1}{2F_{3n-1} - 1}.
\]

Therefore, we obtain

\[
\frac{1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{1}{2F_{3n-1} - 1},
\]

which shows that the statement is true when $n$ is even.

We now turn to consider the case where $n$ is odd. If $m$ is even, applying Lemma 31 and Lemma 33, we can deduce that

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{-1}{2F_{3n-1} + 1} + \frac{1}{2F_{3mn+2}} - \sum_{k=n}^{mn} s_3(k) > \frac{-1}{2F_{3n-1}},
\]

and

\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{-1}{2F_{3n-1} + 1} - \sum_{k=n+2}^{mn} s_4(k) - \left( s_4(n) + s_4(n+1) - \frac{1}{2F_{3mn+2} + 1} \right)
\]

\[
\leq \frac{-1}{2F_{3n-1} + 1} - \sum_{k=n+2}^{mn} s_4(k) - \left( s_4(n) + s_4(n+1) - \frac{1}{2F_{6n+2} + 1} \right)
\]

\[
< \frac{-1}{2F_{3n-1} + 1}.
\]
Thus, if \( n \) is odd and \( m \) is even, we have
\[
\frac{-1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{-1}{2F_{3n-1} + 1}.
\]

If \( m \) is odd, then Lemma 31 and Lemma 32 implies that
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{-1}{2F_{3n-1}} + \frac{1}{2F_{3mn+2}} - \sum_{k=n+2}^{mn-1} s_3(k) - (s_3(n) + s_3(n+1) + s_3(mn)) > \frac{-1}{2F_{3n-1}}.
\]

And it follows from Lemma 33 that
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = -\frac{1}{2F_{3n-1} + 1} - \sum_{k=n}^{mn-1} s_4(k) - \left( s_4(mn) - \frac{1}{2F_{3mn+2} + 1} \right)
\]
\[
= -\frac{1}{2F_{3n-1} + 1} - \sum_{k=n}^{mn-1} s_4(k) - \left( \frac{1}{F_{3mn}} - \frac{1}{2F_{3mn+1} + 1} \right)
\]
\[
< -\frac{1}{2F_{3n-1} + 1}.
\]

Thus, if \( n \) and \( m \) are both odd, then
\[
\frac{-1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{-1}{2F_{3n-1} + 1}
\]
also holds. Hence, the statement is true when \( n \) is odd.

\[\square\]

**Theorem 35.** If \( n \geq 2 \), then
\[
\left[ \frac{2n}{\left( \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} \right)^{-1}} \right] = \begin{cases} 
2F_{3n} - 1, & \text{if } n \text{ is even;} \\
-2F_{3n} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Theorem 36.** If \( n \geq 1 \) and \( m \geq 3 \), then
\[
\left[ \frac{mn}{\left( \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} \right)^{-1}} \right] = \begin{cases} 
2F_{3n}, & \text{if } n \text{ is even;} \\
-2F_{3n}, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Theorem 37.** If \( n \geq 1 \) and \( m \geq 2 \), then
\[
\left[ \frac{mn}{\left( \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+2}} \right)^{-1}} \right] = \begin{cases} 
2F_{3n+1} - 1, & \text{if } n \text{ is even;} \\
-2F_{3n+1} - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Remark 38.** We will prove Theorem 35 and Theorem 36 in detail in the next section. The proof of Theorem 37 is very similar to that of Theorem 34, thus omitted here.
5 Proof of Theorem 35 and Theorem 36

We begin with introducing the following auxiliary functions:

\[
\begin{align*}
t_1(n) &= \frac{1}{2F_{3n}} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3}}, \\
t_2(n) &= \frac{1}{2F_{3n} - 1} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3} - 1}, \\
t_3(n) &= \frac{1}{2F_{3n} + 1} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3} + 1}, \\
t_4(n) &= -\frac{1}{2F_{3n}} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3}}, \\
t_5(n) &= -\frac{1}{2F_{3n} + 1} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3} + 1}, \\
t_6(n) &= -\frac{1}{2F_{3n} - 1} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3} - 1}.
\end{align*}
\]

It is straightforward to check that each \(t_i(n)\) is positive if \(n\) is odd, and negative otherwise.

**Lemma 39.** For all \(n \geq 1\), we have \(t_1(n) + t_1(n + 1) > 0\) and

\[t_1(n) + t_1(n + 1) > t_1(n + 2) + t_1(n + 3)\]

**Proof.** If \(n\) is odd, we have

\[
t_1(n) + t_1(n + 1) = \left(\frac{1}{2F_{3n}} - \frac{1}{2F_{3n+6}}\right) + \left(\frac{1}{F_{3n+1}} - \frac{1}{F_{3n+4}}\right) = \frac{2F_{3n+3}}{F_{3n}F_{3n+6}} + \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} > 0.
\]

Since

\[
\frac{F_{3n+3}}{F_{3n}F_{3n+6}} > \frac{F_{3n+9}}{F_{3n+6}F_{3n+12}}, \quad \frac{F_{3n+2}}{F_{3n+1}F_{3n+4}} > \frac{F_{3n+8}}{F_{3n+7}F_{3n+10}},
\]

we can conclude that \(t_1(n) + t_1(n + 1) > t_1(n + 2) + t_1(n + 3)\).
Now we consider the case where \( n \) is even. Applying Lemma 5 repeatedly, we have

\[
t_1(n) + t_1(n + 1) = \left( \frac{1}{F_{3n}} - \frac{1}{2F_{3n+6}} \right) - \left( \frac{1}{F_{3n+1}} - \frac{1}{F_{3n+4}} \right)
\]

\[
= \frac{2F_{3n+3}}{F_{3n}F_{3n+6}} - \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}
\]

\[
= 2 \cdot \frac{F_{3n+1}F_{3n+3}F_{3n+4} - F_{3n}F_{3n+2}F_{3n+6}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
= 2 \cdot \frac{F_{3n+1}F_{3n+2}F_{3n+3} + F_{3n+1}F_{3n+4}^2}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
- 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+4} + F_{3n}F_{3n+2}F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
= 2 \cdot \frac{F_{3n+2}(F_{3n+1}F_{3n+3} - F_{3n}F_{3n+4}) + F_{3n+1}(F_{3n+1}F_{3n+5} - 1)}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
- 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
= 2 \cdot \frac{2F_{3n+2} - F_{3n+1}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
= 2 \cdot \frac{F_{3n} + F_{3n+2} + F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
> 0.
\]

In addition, it is easy to see that \( F_{3n} + F_{3n+2} + F_{3n+5} = 3F_{3n} + 3F_{3n+1} + F_{3n+4} \), thus

\[
t_1(n) + t_1(n + 1) = \frac{6}{F_{3n+1}F_{3n+4}F_{3n+6}} + \frac{6}{F_{3n}F_{3n+4}F_{3n+6}} + \frac{2}{F_{3n}F_{3n+1}F_{3n+6}},
\]

which decreases as \( n \) grows.

**Lemma 40.** For all \( n \geq 1 \), we have

\[
2F_{3n+3} > F_nF_{n+1}F_{n+6}.
\]
Proof. Applying Lemma 3 repeatedly, we obtain

\[
F_{3n+3} = F_nF_{2n+2} + F_{n+1}F_{2n+3}
\]
\[
= F_n(F_nF_{n+1} + F_{n+1}F_{n+2}) + F_{n+1}(F_nF_{n+2} + F_{n+1}F_{n+3})
\]
\[
= F_nF_{n+1}(F_n + 2F_{n+2}) + F_{n+1}^2(F_n + 2F_n)
\]
\[
= F_nF_{n+1}(F_n + 2F_{n+1} + 2F_{n+2}) + 2F_{n+1}^3
\]
\[
> F_nF_{n+1}(3F_{n+2} + 2F_{n+1})
\]
\[
= F_nF_{n+1}F_{n+5}.
\]

Therefore,

\[
2F_{3n+3} - F_nF_{n+1}F_{n+6} > 2F_nF_{n+1}F_{n+5} - F_nF_{n+1}F_{n+6} = F_nF_{n+1}(2F_{n+5} - F_{n+6}) > 0,
\]

which completes the proof.

Lemma 41. If \( n \geq 1 \) and \( m \geq 3 \), then

\[
t_1(n) + t_1(n + 1) + t_1(mn) > 0.
\]

Proof. If \( mn \) is odd, then the result follows from Lemma 39 and the fact \( t_1(mn) > 0 \). Now we assume that \( mn \) is even. It follows from Lemma 5 that \( F_{3mn}F_{3mn+1} = F_{3mn-2}F_{3mn+3} + 2 \), from which we get

\[
t_1(mn) = \frac{1}{2F_{3mn}} - \frac{1}{F_{3mn+1}} - \frac{1}{2F_{3mn+3}}
\]
\[
= - \frac{F_{3mn-2}}{2(F_{3mn-2}F_{3mn+3} + 2)} - \frac{1}{2F_{3mn+3}}
\]
\[
> - \frac{F_{3mn-2}}{2F_{3mn-2}F_{3mn+3}} - \frac{1}{2F_{3mn+3}}
\]
\[
= - \frac{1}{F_{3mn+3}}
\]
\[
\geq - \frac{1}{F_{9n+3}}.
\]

On the other hand, it follows from the proof of Lemma 39 that

\[
t_1(n) + t_1(n + 1) > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}}.
\]

Now we arrive at

\[
t_1(n) + t_1(n + 1) + t_1(mn) > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}} - \frac{1}{F_{9n+3}} > 0,
\]

where the last inequality follows from Lemma 40. □
Lemma 42. For all \( n \geq 2 \), we have
\[
F_{2n}F_{2n+1} - F_{n+1}F_{n+2}F_{2n-2} < 0.
\]

Proof. It follows from Lemma 4 and Lemma 5 respectively that
\[
F_{n+2}F_{n+3} - F_nF_{n+1} = F_{2n+3},
\]
\[
F_{n+1}F_{n+4} - F_{n+2}F_{n+3} = (-1)^n,
\]
from which we can deduce that
\[
F_{n+1}F_{n+4} = F_nF_{n+1} + F_{2n+3} + (-1)^n > F_{2n+3} + 2.
\]
Therefore,
\[
F_{2n}F_{2n+1} - F_{n+1}F_{n+4}F_{2n-2} < F_{2n}F_{2n+1} - (F_{2n+3} + 2)F_{2n-2}
= (F_{2n}F_{2n+1} - F_{2n-2}F_{2n+3}) - 2F_{2n-2}
= 2 - 2F_{2n-2}
\leq 0,
\]
where the last equality follows from Lemma 5. \( \square \)

Lemma 43. If \( n \geq 2 \) is even, then
\[
\sum_{k=n}^{2n} t_1(k) + \frac{1}{2F_{6n-3}} < 0.
\]

Proof. From the proof of Lemma 39 we know that if \( n \) is even, then
\[
t_1(n) + t_1(n + 1) = 2 \cdot F_{3n} + F_{3n+2} + F_{3n+5} < \frac{2F_{3n+6}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} = \frac{2}{F_{3n}F_{3n+1}F_{3n+4}}.
\]
Applying Lemma 39 again and the above inequality, we have
\[
\sum_{k=n}^{2n} t_1(k) + \frac{1}{2F_{6n-3}} = \sum_{k=n}^{2n-1} t_1(k) + \left( t_1(2n) + \frac{1}{2F_{6n+3}} \right)
< \frac{2}{F_{3n}F_{3n+1}F_{3n+4}} \cdot \frac{n}{2} + \left( \frac{1}{2F_{6n}} - \frac{1}{F_{6n+1}} \right)
= \frac{n}{F_{3n}F_{3n+1}F_{3n+4}} - \frac{F_{6n-2}}{2F_{6n}F_{6n+1}}
< \frac{1}{2F_{3n+1}F_{3n+4}} - \frac{F_{6n-2}}{2F_{6n}F_{6n+1}}
< 0,
\]
where the last inequality follows from Lemma 42. \( \square \)
Lemma 44. For all \( n \geq 1 \), we have
\[
t_2(n) + t_2(n + 1) > 0.
\]

Proof. It is easy to see that the result is true when \( n \) is odd. So we assume that \( n \) is even. It follows from the definition of \( t_2(n) \) that
\[
t_2(n) + t_2(n + 1) = \left( \frac{1}{2F_{3n} - 1} - \frac{1}{2F_{3n+6} - 1} \right) - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}}
\]
\[
= \frac{2F_{3n+6} - 2F_{3n}}{(2F_{3n} - 1)(2F_{3n+6} - 1)} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}}
\]
\[
> \frac{F_{3n+6} - F_{3n}}{2F_{3n}F_{3n+6}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}}
\]
\[
= \frac{1}{2F_{3n}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} - \frac{1}{2F_{3n+6}}
\]
\[
= t_1(n) + t_1(n + 1)
\]
\[
> 0,
\]
where the last inequality follows from the proof of Lemma 39.

Lemma 45. If \( n \geq 1 \) and \( m \geq 2 \), then
\[
t_2(n) + t_2(n + 1) + t_2(mn) > 0.
\]

Proof. If \( mn \) is odd, then the result follows from Lemma 44 and the fact \( t_2(mn) > 0 \). Thus we assume that \( mn \) is even in the rest. Applying the argument in the proof of Lemma 44 and Lemma 41, we can easily obtain
\[
t_2(mn) > t_1(mn) > -\frac{1}{F_{3mn+3}} \geq -\frac{1}{F_{6n+3}}.
\]
If \( n \) is odd, we have
\[
t_2(n) + t_2(n + 1) > \frac{1}{F_{3n+1}} - \frac{1}{F_{3n+4}} = \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} > \frac{2}{(2F_{3n} - 1)F_{3n+4}}.
\]
If \( n \) is even, then from the proof of Lemma 44 and Lemma 39 we know that

\[
\begin{align*}
t_2(n) + t_2(n + 1) &> \frac{F_{3n+6} - F_{3n}}{(2F_{3n} - 1)F_{3n+6}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} \\
&= \frac{4F_{3n+3}}{(2F_{3n} - 1)F_{3n+6}} - \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} \\
&= \frac{4(F_{3n+1}F_{3n+3}F_{3n+4} - F_{3n}F_{3n+2}F_{3n+6}) + 2F_{3n+2}F_{3n+6}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}F_{3n+6}} \\
&> \frac{2F_{3n+2}F_{3n+6}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}F_{3n+6}} \\
&> \frac{2}{(2F_{3n} - 1)F_{3n+4}}.
\end{align*}
\]

Therefore, we always have

\[
t_2(n) + t_2(n + 1) > \frac{2}{(2F_{3n} - 1)F_{3n+4}},
\]

from which we get

\[
t_2(n) + t_2(n + 1) + t_2(mn) > \frac{2}{(2F_{3n} - 1)F_{3n+4}} - \frac{1}{F_{6n+3}} > 0,
\]

where the last inequality follows from the fact \( F_{6n+3} = F_{3n-1}F_{3n+3} + F_{3n}F_{3n+4} \).

Lemma 46. If \( n \geq 2 \) is even, then

\[
t_3(n) + t_3(n + 1) < 0.
\]
Proof. Applying the analysis in the proof of Lemma 39, we can deduce that

\[ t_3(n) + t_3(n + 1) = \left( \frac{1}{2F_{3n} + 1} - \frac{1}{2F_{3n+6} + 1} \right) - \left( \frac{1}{F_{3n+1} - \frac{1}{F_{3n+4}}} \right) \]

\[ = \frac{2F_{3n+6} - 2F_{3n}}{(2F_{3n} + 1)(2F_{3n+6} + 1)} - \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} \]

\[ < \frac{F_{3n+6} - F_{3n}}{(2F_{3n} + 1)F_{3n+6} - \frac{2F_{3n+2}}{2F_{3n+1}F_{3n+4}}} \]

\[ = \frac{4(F_{3n+1}F_{3n} + 3F_{3n+4} - F_{3n}F_{3n+2}F_{3n+6}) - 2F_{3n+2}F_{3n+6}}{(2F_{3n} + 1)F_{3n+1}F_{3n+4}F_{3n+6}} \]

\[ = \frac{4(F_{3n} + F_{3n+2} + F_{3n+5}) - 2F_{3n+2}F_{3n+6}}{(2F_{3n} + 1)F_{3n+1}F_{3n+4}F_{3n+6}} \]

\[ < \frac{4F_{3n+6} - 2F_{3n+2}F_{3n+6}}{(2F_{3n} + 1)F_{3n+1}F_{3n+4}F_{3n+6}} \]

\[ < 0. \]

The proof is completed. 

\[ \square \]

**Lemma 47.** If \( n \geq 1 \) is odd, then

\[ t_4(n) + t_4(n + 1) > \frac{1}{2F_{9n+3}}. \]

**Proof.** Applying similar arguments in the proof of Lemma 39, we obtain that if \( n \) is odd,

\[ t_4(n) + t_4(n + 1) = 2 \cdot \frac{F_{3n} + F_{3n+2} + F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}}. \]

It follows from Lemma 40 that

\[ \frac{1}{F_{3n}F_{3n+1}F_{3n+6}} > \frac{1}{2F_{9n+3}}. \]

Combining the above two inequalities yields the desired result. \( \square \)

**Lemma 48.** For all \( n \geq 2 \), we have

\[ \sum_{k=n}^{2n} t_4(k) < \frac{1}{2F_{6n+3}}. \]
Proof. If \( n \) is even, it is easy to see that \( t_4(n) + t_4(n + 1) < 0 \). Thus,

\[
\sum_{k=n}^{2n} t_4(k) = \sum_{k=n}^{2n-1} t_4(k) + t_4(2n) < 0 < \frac{1}{2F_{6n+3}}.
\]

If \( n \) is odd,

\[
t_4(n) + t_4(n + 1) = 2 \cdot \frac{F_{3n} + F_{3n+2} + F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} < \frac{2}{F_{3n}F_{3n+1}F_{3n+4}},
\]

which implies that

\[
\sum_{k=n}^{2n} t_4(k) < \frac{2}{F_{3n}F_{3n+1}F_{3n+4}} \cdot \frac{n}{2} < \frac{1}{2F_{3n+1}F_{3n+4}} < \frac{1}{2F_{6n+3}},
\]

where the last inequality follows from that for \( n \geq 1, \)

\[
F_{3n+1}F_{3n+4} = F_{3n-1}F_{3n+4} + F_{3n}F_{3n+4} > F_{3n-1}F_{3n+3} + F_{3n}F_{3n+1} = F_{6n+3}.
\]

This completes the proof.

Lemma 49. If \( n \geq 1 \) is odd, then

\[
t_5(n) + t_5(n + 1) > \frac{1}{2F_{6n+3} + 1}.
\]

Proof. Imitating the proof of Lemma 46, we can easily obtain

\[
t_5(n) + t_5(n + 1) > \frac{4(F_{3n} + F_{3n+2} + F_{3n+5}) + 2F_{3n+2}F_{3n+6}}{(2F_{3n} + 1)F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
> \frac{2F_{3n+2}F_{3n+6}}{(2F_{3n} + 1)F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
> \frac{1}{F_{3n+1}F_{3n+4}}.
\]

In addition, we have

\[
2F_{2n+3} = 2F_{n-1}F_{n+3} + 2F_nF_{n+4} > F_{n+1}F_{n+4}.
\]

Combining the above two inequalities together yields the desired result.

Lemma 50. For all \( n \geq 1 \), we have

\[
t_6(n) + t_6(n + 1) < 0.
\]
Proof. It is clear that the result holds if \( n \) is even. Now we assume that \( n \) is odd. Applying the telescoping technique in the proof of Lemma 45 and the similar analysis in the proof of Lemma 39, we obtain

\[
t_6(n) + t_6(n + 1) < - \left\{ \frac{4F_{3n+3}}{(2F_{3n} - 1)F_{3n+6}} - \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} \right\}
\]

\[
= - \frac{4(F_{3n+1}F_{3n+3}F_{3n+4} - F_{3n}F_{3n+2}F_{3n+6}) + 2F_{3n+2}F_{3n+6}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
= - \frac{-4(F_{3n} + F_{3n+2} + F_{3n+5}) + 2F_{3n+2}F_{3n+6}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
< \frac{4F_{3n+6} - 2F_{3n+2}F_{3n+6}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}F_{3n+6}}
\]

\[
< 0,
\]

which completes the proof.

\[\square\]

**Lemma 51.** If \( n \geq 1 \) and \( m \geq 2 \), we have

\[
t_6(n) + t_6(n + 1) + t_6(mn) < 0.
\]

*Proof.* If \( mn \) is even, then the result follows from Lemma 50 and the fact \( t_6(mn) < 0 \), so we assume that \( mn \) is odd in the rest. Now we have

\[
t_6(mn) = \frac{-1}{2F_{3mn} - 1} + \frac{1}{F_{3mn+1}} + \frac{1}{2F_{3mn+3} - 1}
\]

\[
= \frac{-4F_{3mn+1}}{(2F_{3mn} - 1)(2F_{3mn+3} - 1)} + \frac{1}{F_{3mn+1}}
\]

\[
< \frac{-2F_{3mn+1}}{(2F_{3mn} - 1)F_{3mn+3}} + \frac{1}{F_{3mn+1}}
\]

\[
= \frac{-2F_{3mn+1}^2 + 2F_{3mn}F_{3mn+3} - F_{3mn+3}}{(2F_{3mn} - 1)F_{3mn+1}F_{3mn+3}}
\]

\[
= \frac{-2(F_{3mn+1}^2 - F_{3mn}F_{3mn+2}) + 2F_{3mn}F_{3mn+1} - F_{3mn+1} - F_{3mn+2}}{(2F_{3mn} - 1)F_{3mn+1}F_{3mn+3}}
\]

\[
= \frac{2 + (2F_{3mn} - 1)F_{3mn+1} - F_{3mn+2}}{(2F_{3mn} - 1)F_{3mn+1}F_{3mn+3}}
\]

\[
< \frac{(2F_{3mn} - 1)F_{3mn+1}}{(2F_{3mn} - 1)F_{3mn+1}F_{3mn+3}}
\]

\[
= \frac{1}{F_{3mn+3}}.
\]
Since $mn$ is odd, we must have that $n$ is odd and $m \geq 3$. Therefore,
\[ t_6(mn) < \frac{1}{F_{9n+3}}. \]

It follows from the proof of Lemma 50 that if $n$ is odd,
\[ t_6(n) + t_6(n + 1) < \frac{4 - 2F_{3n+2}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}} < \frac{2 - F_{3n+2}}{F_{3n}F_{3n+1}F_{3n+4}} \]
\[ < - \frac{1}{F_{3n}F_{3n+1}F_{3n+4}}. \]

Now we arrive at
\[ t_6(n) + t_6(n + 1) + t_6(mn) < \frac{1}{F_{9n+3}} - \frac{1}{F_{3n}F_{3n+1}F_{3n+4}}. \]

Employing Lemma 3, we easily see that $F_{3n+3} > F_{2n}F_{n+4}$ and $F_{2n} > F_nF_{n+1}$, which implies
\[ F_{9n+3} > F_{3n}F_{3n+1}F_{3n+4}. \]

Therefore,
\[ t_6(n) + t_6(n + 1) + t_6(mn) < \frac{1}{F_{9n+3}} - \frac{1}{F_{3n}F_{3n+1}F_{3n+4}} < 0. \]

The proof is completed. \qed

**Proof of Theorem 35.** We first consider the case where $n$ is even. Applying Lemma 43, we have
\[ \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n}} - \frac{1}{2F_{6n+3}} - \sum_{k=n}^{2n} t_1(k) > \frac{1}{2F_{3n}}. \]

It follows from Lemma 44 and Lemma 45 that
\[ \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n} - 1} - \frac{1}{2F_{6n+3} - 1} - (t_2(n) + t_2(n + 1) + t_2(2n)) - \sum_{k=n+2}^{2n-1} t_2(k) < \frac{1}{2F_{3n} - 1}. \]

Therefore, we have
\[ \frac{1}{2F_{3n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} < \frac{1}{2F_{3n} - 1}, \]

which means that the result holds when $n$ is even.

We now turn to consider the case where $n$ is odd. From Lemma 48, we know that
\[ \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n}} + \frac{1}{2F_{6n+3}} - \sum_{k=n}^{2n} t_4(k) > \frac{-1}{2F_{3n}}. \]
With the help of Lemma 49, it is easy to see that
\[
\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n} + 1} \left( t_5(n) + t_5(n+1) - \frac{1}{2F_{6n+3} + 1} \right) - \sum_{k=n+2}^{2n} t_5(k) < \frac{-1}{2F_{3n} + 1}.
\]
Thus, we obtain
\[
\frac{-1}{2F_{3n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} < \frac{-1}{2F_{3n} + 1},
\]
which yields the desired identity.

\[\square\]

**Proof of Theorem 36.** We first consider the case where \( n \) is even. Applying Lemma 39 and Lemma 41, we see
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n}} - \frac{1}{2F_{3mn+3}} - \sum_{k=n+2}^{mn-1} t_1(k) - (t_1(n) + t_1(n+1) + t_1(mn)) < \frac{1}{2F_{3n}}.
\]
It follows from Lemma 46 that
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n} + 1} - \sum_{k=n}^{mn-1} t_5(k) - \left( t_5(mn) + \frac{1}{2F_{3mn+3} + 1} \right)
\]
\[
= \frac{1}{2F_{3n} + 1} - \sum_{k=n}^{mn-1} t_5(k) - \left( \frac{1}{2F_{3mn} + 1} - \frac{1}{2F_{3mn+1}} \right)
\]
\[
> \frac{1}{2F_{3n} + 1}.
\]
Therefore, we obtain
\[
\frac{1}{2F_{3n} + 1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < \frac{1}{2F_{3n}},
\]
which shows that the statement is true when \( n \) is even.

Now we turn to consider the case where \( n \) is odd. Lemma 47 tells us that
\[t_4(n) + t_4(n+1) > 0.\]
Hence if \( mn \) is odd,
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n}} - \sum_{k=n}^{mn-1} t_4(k) - \left( t_4(mn) - \frac{1}{2F_{3mn+3}} \right)
\]
\[
= \frac{-1}{2F_{3n}} - \sum_{k=n}^{mn-1} t_4(k) - \left( \frac{1}{2F_{3mn+1}} - \frac{1}{2F_{3mn}} \right)
\]
\[
< \frac{-1}{2F_{3n}}.
\]

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And it follows from Lemma 50 and Lemma 51 that
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = -\frac{1}{2F_{3n} - 1} + \frac{1}{2F_{3mn+3} - 1} - \sum_{k=n+2}^{mn-1} t_6(k) - (t_6(n) + t_6(n + 1) + t_6(mn))
\]
\[
> -\frac{1}{2F_{3n} - 1}.
\]

Therefore, we have
\[
-\frac{1}{2F_{3n} - 1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < -\frac{1}{2F_{3n}}.
\]

If \(mn\) is even, then Lemma 47 implies that
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = -\frac{1}{2F_{3n} - 1} - \sum_{k=n+2}^{mn} t_4(k) - \left( t_4(n) + t_4(n + 1) - \frac{1}{2F_{3mn+3}} \right)
\]
\[
< -\frac{1}{2F_{3n} - 1} - \sum_{k=n+2}^{mn} t_4(k) - \left( t_4(n) + t_4(n + 1) - \frac{1}{2F_{9n+3}} \right)
\]
\[
< -\frac{1}{2F_{3n}},
\]
and from Lemma 50 we obtain
\[
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = -\frac{1}{2F_{3n} - 1} + \frac{1}{2F_{3mn+3} - 1} - \sum_{k=n}^{mn} t_6(k) > -\frac{1}{2F_{3n} - 1}.
\]

Hence, we also have
\[
-\frac{1}{2F_{3n} - 1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < -\frac{1}{2F_{3n}}.
\]

which yields the desired identity.

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References


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