Jacobsthal and Jacobsthal-Lucas Numbers and Sums Introduced by Jacobsthal and Tverberg

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Abstract
We study the sums introduced by Jacobsthal and Tverberg and show that the extreme values of the sums are connected with Jacobsthal and Jacobsthal-Lucas numbers.

1 Introduction
Let \( a, b \in \mathbb{Z} \) and \( m \in \mathbb{Z}^+ \). In 1957, Jacobsthal [4] introduced the sums of the form

\[
S_{a,b,m}(K) = \sum_{k=0}^{K} f_{a,b,m}(k),
\]

where

\[
f_{a,b,m}(k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor.
\]
In the above equation and throughout this article, unless stated otherwise, \( k \) is an integer and \( K \) is a nonnegative integer. So we can consider \( f_{a,b;m} \) and \( S_{a,b;m} \) as functions of \( k \) and \( K \) defined on \( \mathbb{Z} \) and on \( \mathbb{N} \cup \{0\} \), respectively.

These sums are also studied by Carlitz [1, 2], Grimson [3] and recently by Tverberg [6]. In addition, Tverberg [6] extends the definition of \( f_{a,b;m}(k) \) and \( S_{a,b;m}(K) \) to the following form.

**Definition 1.** Let \( m \) and \( \ell \) be positive integers and let \( C \) be a multiset of \( \ell \) integers \( a_1, a_2, \ldots, a_\ell \), i.e., \( a_i = a_j \) is allowed for some \( i \neq j \). Define \( f_{C;m} : \mathbb{Z} \to \mathbb{Z} \) and \( S_{C;m} : \mathbb{N} \cup \{0\} \to \mathbb{Z} \) by

\[
f_{C;m}(k) = \sum_{T \subseteq [1, \ell]} (-1)^{\ell - |T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor,
\]

\[
S_{C;m}(K) = \sum_{k=0}^{K} f_{C;m}(k).
\]

We sometimes write \( f_{a_1,a_2,\ldots,a_\ell;m}(k) \) and \( S_{a_1,a_2,\ldots,a_\ell;m}(K) \) instead of \( f_{C;m}(k) \) and \( S_{C;m}(K) \), respectively. The set \([1, \ell]\) appearing in the sum defining \( f \) is \( \{1, 2, 3, \ldots, \ell\} \) and if \( T = \emptyset \), then \( \sum_{i \in T} a_i \) is defined to be zero.

For example, if \( C = \{a, b\} \), then \( f_{C;m}(k) \) given in Definition 1 is the same as \( f_{a,b;m}(k) \) given in (1), and if \( C = \{a_1, a_2, a_3\} \), then \( f_{C;m}(k) \) is

\[
f_{a_1,a_2,a_3;m}(k) = \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor.
\]

Jacobsthal [4] shows that for any \( K \in \mathbb{N} \cup \{0\} \), we have

\[
0 \leq S_{a,b;m}(K) \leq \left\lfloor \frac{m}{2} \right\rfloor, \tag{2}
\]

which is a sharp inequality, that is, the lower bound 0 is actually the minimum value and the upper bound \( \left\lfloor \frac{m}{2} \right\rfloor \) is the maximum value of \( S_{a,b;m}(K) \). Tverberg [6] proves (2) in a different way and he also gives the extreme values of \( S_{a_1,a_2,a_3;m}(K) \) without proof. Nevertheless, the extreme values of \( f_{a_1,a_2,\ldots,a_\ell;m}(k) \) (for \( \ell \geq 2 \)) and \( S_{a_1,a_2,\ldots,a_\ell;m}(K) \) (for \( \ell \geq 4 \)) have not been calculated.

In this article, we calculate the extreme values of \( f_{a_1,a_2,\ldots,a_\ell;m}(k) \) for all \( \ell \geq 2 \) (see Theorem 8). We also introduce the function \( g \) in Definition 2, give its connection with \( f_{a_1,a_2,\ldots,a_\ell;m}(k) \), and obtain its extreme values (see Proposition 3 and Theorem 4). Furthermore, we obtain the minimum value of \( S_{a_1,a_2,\ldots,a_\ell;m}(K) \) when \( \ell \) is odd and the maximum value of \( S_{a_1,a_2,\ldots,a_\ell;m}(K) \) when \( \ell \) is even (see Theorem 9).
The reader will see that the extreme values of the functions $g$ and $f_{a_1,a_2,\ldots,a_{\ell};m}(k)$ are connected with Jacobsthal numbers $J_n$ and Jacobsthal-Lucas numbers $j_n$ defined, respectively, by the recurrence relations

\[
\begin{align*}
J_0 &= 0, & J_1 &= 1, & J_n &= J_{n-1} + 2J_{n-2} & \text{for } n \geq 2, \\
j_0 &= 2, & j_1 &= 1, & j_n &= j_{n-1} + 2j_{n-2} & \text{for } n \geq 2.
\end{align*}
\]

The sequences $(J_n)_{n \geq 0}$ and $(j_n)_{n \geq 0}$ are, respectively, A001045 and A014551 in the OEIS [5].

The function $g$ is defined as follows:

**Definition 2.** Let $g : \mathbb{R}^n \to \mathbb{Z}$ be given by

\[
g(x_1, x_2, x_3, \ldots, x_n) = \sum_{1 \leq i \leq n} \lfloor x_i \rfloor - \sum_{1 \leq i_1 < i_2 \leq n} \lfloor x_{i_1} + x_{i_2} \rfloor + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \lfloor x_{i_1} + x_{i_2} + x_{i_3} \rfloor - \cdots + (-1)^{n-1} \lfloor x_1 + x_2 + x_3 + \cdots + x_n \rfloor.
\]

In other words,

\[
g(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left| \sum_{i \in T} x_i \right|.
\]

## 2 Main results

We begin this section by giving a relation between the functions $f$ and $g$. Then we give the extreme values of $g$ and $f$ and their connection with Jacobsthal and Jacobsthal-Lucas numbers.

**Proposition 3.** For each $\ell \geq 2$, we have

(i) $f_{a_1,a_2,\ldots,a_{\ell};m}(0) = (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m} \right)$,

(ii) $f_{a_1,a_2,\ldots,a_{\ell};m}(k) = (-1)^{\ell} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m}, \frac{k}{m} \right) + (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m} \right)$.

**Proof.** This follows easily from the definitions of $f$ and $g$ but we give a proof for completeness. We have

\[
f_{a_1,a_2,\ldots,a_{\ell};m}(0) = \sum_{T \subseteq [1,\ell]} (-1)^{|T|} \left| \sum_{i \in T} \left( \frac{a_i}{m} \right) \right|
\]

\[
= \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{|T|} \left| \sum_{i \in T} \left( \frac{a_i}{m} \right) \right|
\]

\[
= (-1)^{\ell-1} \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{1-|T|} \left| \sum_{i \in T} \left( \frac{a_i}{m} \right) \right|
\]

\[
= (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m} \right).
\]
Next let \( a_{\ell+1} = k \). Then we obtain
\[
(-1)^{\ell} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m}, \frac{k}{m} \right) + (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_{\ell}}{m} \right)
\]
\[
= (-1)^{\ell} \sum_{T \subseteq [\ell+1]} (-1)^{|T|-1} \left[ \sum_{i \in T} \left( \frac{a_i}{m} \right) - \sum_{\emptyset \neq T' \subseteq [\ell]} (-1)^{|T'|-1} \left[ \sum_{i \in T'} \left( \frac{a_i}{m} \right) \right] \right]
\]
\[
= (-1)^{\ell} \sum_{T \subseteq [\ell+1]} (-1)^{|T|-1} \left[ \sum_{i \in T} \left( \frac{k + \sum_{i \in T} a_i}{m} \right) \right]
\]
\[
= f_{a_1, a_2, \ldots, a_{\ell}; m}(k).
\]

\( \square \)

**Theorem 4.** For each \( n \geq 2 \), the function \( g \) given in Definition 2 has maximum value \( 2^{n-1} - 1 \) and minimum value \( -2^{n-2} \). The minimum occurs at least when \( x_k = \frac{1}{2} \) for every \( 1 \leq k \leq n \). The maximum occurs at least when \( x_k = \frac{1}{2} - \frac{1}{m} \) for every \( 1 \leq k \leq n \).

**Proof.** If \( n = 2 \), then the result is a well-known inequality
\[
-1 \leq |x| + |y| - |x+y| \leq 0,
\]
which holds for all \( x, y \in \mathbb{R} \). The inequality (3) is sharp: if \( x = y = \frac{1}{2} \) the left inequality in (3) becomes equality, and if \( x = y = \frac{1}{3} \) the right inequality in (3) becomes equality. The result when \( n \geq 3 \) is obtained from the case \( n = 2 \) and a careful selection of pairs. For illustration purpose, we first give a proof for the case \( n = 3 \) and \( n = 4 \). Recall that
\[
g(x_1, x_2, x_3) = [x_1] + [x_2] + [x_3] - [x_1 + x_2] - [x_1 + x_3] - [x_2 + x_3] + [x_1 + x_2 + x_3].
\]
We obtain by (3) that
\[
0 \leq [x_1 + x_2 + x_3] - [x_1 + x_2] - [x_3] \leq 1,
\]
\[
-1 \leq [x_2 + x_3] + [x_2] + [x_3] \leq 0,
\]
\[
-1 \leq -[x_1 + x_3] + [x_1] + [x_3] \leq 0.
\]
Summing (4), (5), and (6), the middle terms give \( g(x_1, x_2, x_3) \). Then \(-2 \leq g(x_1, x_2, x_3) \leq 1\).

Next we consider
\[
g(x_1, x_2, x_3, x_4) = [x_1] + [x_2] + [x_3] + [x_4] - [x_1 + x_2] - [x_1 + x_3] - [x_1 + x_4]
\]
\[
- [x_2 + x_3] - [x_2 + x_4] - [x_3 + x_4] + [x_1 + x_2 + x_3] + [x_1 + x_2 + x_4]
\]
\[
+ [x_1 + x_3 + x_4] + [x_2 + x_3 + x_4] - [x_1 + x_2 + x_3 + x_4].
\]
Again, we obtain by (3) the following inequalities:

\[-1 \leq -[x_1 + x_2 + x_3 + x_4] + [x_1 + x_2 + x_3] + [x_4] \leq 0, \tag{7}\]

\[0 \leq [x_1 + x_2 + x_4] - [x_1 + x_2] - [x_4] \leq 1, \tag{8}\]

\[0 \leq [x_1 + x_3 + x_4] - [x_1 + x_3] - [x_4] \leq 1, \tag{9}\]

\[0 \leq [x_2 + x_3 + x_4] - [x_2 + x_3] - [x_4] \leq 1, \tag{10}\]

\[-1 \leq -[x_1 + x_4] + [x_1] + [x_4] \leq 0, \tag{11}\]

\[-1 \leq -[x_2 + x_4] + [x_2] + [x_4] \leq 0, \tag{12}\]

\[-1 \leq -[x_3 + x_4] + [x_3] + [x_4] \leq 0. \tag{13}\]

Summing (7) to (13), we see that \(-4 \leq g(x_1, x_2, x_3, x_4) \leq 3.

Next we prove the general case \(n \geq 5\). The expression of the form \([x_1 + x_2 + \cdots + x_{i_k}]\) will be called a \(k\)-bracket. So for each \(1 \leq k \leq n\), there are \(\binom{n}{k}\) \(k\)-brackets appearing in the sum defining \(g(x_1, x_2, \ldots, x_n)\). We first pair up the \(n\)-bracket with an \((n - 1)\)-bracket and a 1-bracket as follows:

\[s_1 = (-1)^{n-1}[x_1 + x_2 + \cdots + x_n] + (-1)^{n-2}[x_1 + x_2 + \cdots + x_{n-1}] + (-1)^{n-2}[x_n]. \tag{14}\]

Notice that the sign of \([x_n]\) in (14) may or may not be the same as that appearing in the sum defining \(g(x_1, x_2, \ldots, x_n)\) but it is the same as the sign of \([x_1 + x_2 + \cdots + x_{n-1}]\) so that we can apply (3) to obtain the bound for \(s_1\). Next we pair up the remaining \((n - 1)\)-brackets with \((n - 2)\)-brackets and 1-brackets as follows:

\[(-1)^{n-2}[x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}}] + (-1)^{n-3}[x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}] + (-1)^{n-3}[x_{i_{n-1}}], \tag{15}\]

where \(1 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n\). We note again that the sign of \([x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}}]\) and \([x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}]\) in (15) are the same as those appearing in the sum defining \(g(x_1, x_2, \ldots, x_n)\) while the sign of \([x_{i_{n-1}}]\) in (15) may or may not be the same, but we can apply (3) to obtain the bound of (15). Since \([x_1 + x_2 + \cdots + x_{n-1}]\) appears in (14), the term \(x_{i_{n-1}}\) appearing in the \((n - 1)\)-brackets in (15) is always \(x_n\). So in fact (15) is

\[(-1)^{n-2}[x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n] + (-1)^{n-3}[x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}] + (-1)^{n-3}[x_n]. \tag{16}\]
Then we sum (16) over all possibles \(1 \leq i_1 < i_2 < \ldots < i_{n-2} < n\), and call it \(s_2\). That is

\[
s_2 = (-1)^{n-2} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-2} < n} [x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n] + (-1)^{n-3} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-2} < n} [x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}] + (-1)^{n-3} \binom{n-1}{n-2} [x_n].
\]

We continue doing this process as follows. For each \(0 \leq \ell \leq n-1\), let \(a_{\ell}\) be the sum of all \([x_{i_1} + x_{i_2} + \cdots + x_{i_{n-\ell}}]\) with \(1 \leq i_1 < i_2 < \ldots < i_{n-\ell} \leq n\), \(a_{\ell}\) the sum of all such terms with \(i_{n-\ell} = n\), and \(b_{\ell}\) the sum of all such terms with \(i_{n-\ell} < n\). Therefore \(c_{\ell} = a_{\ell} + b_{\ell}\). As usual, the empty sum is defined to be zero, so \(b_0 = 0\). The number of \((n - \ell)\)-brackets appearing in the sum defining \(c_{\ell}\) is \(\binom{n}{n-\ell}\), the number of \((n - \ell)\)-brackets appearing in the sum defining \(a_{\ell}\) is \(\binom{n-1}{n-\ell-1}\), and the number of \((n - \ell)\)-brackets appearing in the sum defining \(b_{\ell}\) is \(\binom{n-1}{n-\ell}\).

In addition, we have

\[
s_1 = (-1)^{n-1} a_0 + (-1)^{n-2} b_1 + (-1)^{n-2} [x_n],
\]

\[
s_2 = (-1)^{n-2} a_1 + (-1)^{n-3} b_2 + (-1)^{n-3} \binom{n-1}{n-2} [x_n].
\]

In general, for each \(1 \leq \ell \leq n-1\), let

\[
s_{\ell} = (-1)^{n-\ell} a_{\ell-1} + (-1)^{n-\ell-1} b_{\ell} + (-1)^{n-\ell-1} \binom{n-1}{n-\ell} [x_n].
\]

Then

\[
\sum_{1 \leq \ell \leq n-1} s_{\ell} = (-1)^{n-1} a_0 + \sum_{2 \leq \ell \leq n-1} (-1)^{n-\ell} a_{\ell-1} + \sum_{1 \leq \ell \leq n-2} (-1)^{n-\ell-1} b_{\ell} + b_{n-1} + [x_n] \sum_{1 \leq \ell \leq n-1} (-1)^{n-\ell-1} \binom{n-1}{n-\ell}.
\]

Recall a well known identity \(\sum_{0 \leq \ell \leq n} (-1)^{\ell} \binom{n}{\ell} = 0\) for all \(n \geq 1\). Therefore the last sum on the right hand side of (17) is

\[
- \sum_{1 \leq \ell \leq n-1} (-1)^{n-\ell} \binom{n-1}{n-\ell} = - \sum_{1 \leq \ell \leq n-1} (-1)^{\ell} \binom{n-1}{\ell} = - \sum_{0 \leq \ell \leq n-1} (-1)^{\ell} \binom{n-1}{\ell} + 1 = 1.
\]

Therefore the last term in (17) is \([x_n]\). Replacing \(\ell\) by \(\ell + 1\) in the first sum on the right hand side of (17), we see that
\[
\sum_{1 \leq \ell \leq n-1} s_\ell = (-1)^{n-1}a_0 + \sum_{1 \leq \ell \leq n-2} (-1)^{n-\ell-1}(a_\ell + b_\ell) + b_{n-1} + \lfloor x_n \rfloor
\]
\[
= (-1)^{n-1}c_0 + \sum_{1 \leq \ell \leq n-2} (-1)^{n-\ell-1}c_\ell + b_{n-1} + \lfloor x_n \rfloor
\]
\[
= (-1)^{n-1}c_0 + \sum_{1 \leq \ell \leq n-2} (-1)^{n-\ell-1}c_\ell + c_{n-1}
\]
\[
= \sum_{0 \leq \ell \leq n-1} (-1)^{n-\ell-1}c_\ell
\]
\[
= g(x_1, x_2, \ldots, x_n),
\]

where (18) can be obtained from the definition of \(c_{n-1}, b_{n-1},\) and \(a_{n-1}\) that
\[
c_{n-1} = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \cdots + \lfloor x_n \rfloor,
\]
\[
b_{n-1} = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \cdots + \lfloor x_{n-1} \rfloor,
\]
\[
a_{n-1} = \lfloor x_n \rfloor, \quad \text{and}
\]
\[
c_{n-1} = a_{n-1} + b_{n-1}.
\]

We apply (3) to (14) to obtain
\[
0 \leq s_1 \leq 1 \text{ if } n \text{ is odd, and } -1 \leq s_1 \leq 0 \text{ if } n \text{ is even.}
\]

Similarly, applying (3) to (16), we see that such sum lies in \([0, 1]\) if \(n\) is even, and lies in \([-1, 0]\) if \(n\) is odd. Therefore
\[
0 \leq s_2 \leq \binom{n-1}{n-2} \text{ if } n \text{ is even, and } -\binom{n-1}{n-2} \leq s_2 \leq 0 \text{ if } n \text{ is odd.}
\]

In general, for each \(1 \leq \ell \leq n-1\), we have
\[
0 \leq s_\ell \leq \binom{n-1}{n-\ell}, \text{ if } n \text{ and } \ell \text{ have the same parity,}
\]
\[
-\binom{n-1}{n-\ell} \leq s_\ell \leq 0, \text{ if } n \text{ and } \ell \text{ have a different parity.}
\]

Since \(g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq \ell \leq n-1} s_\ell\), we obtain, for odd \(n\),
\[
- \sum_{1 \leq \ell \leq n-1, \ell \text{ is even}} \binom{n-1}{n-\ell} \leq g(x_1, x_2, \ldots, x_n) \leq \sum_{1 \leq \ell \leq n-1, \ell \text{ is odd}} \binom{n-1}{n-\ell},
\]

and for even \(n\),
\[
- \sum_{1 \leq \ell \leq n-1, \ell \text{ is odd}} \binom{n-1}{n-\ell} \leq g(x_1, x_2, \ldots, x_n) \leq \sum_{1 \leq \ell \leq n-1, \ell \text{ is even}} \binom{n-1}{n-\ell}.
\]
Recall a well known identity
\[ \sum_{0 \leq k \leq n \atop k \text{ is even}} \binom{n}{k} = \sum_{0 \leq k \leq n \atop k \text{ is odd}} \binom{n}{k} = 2^{n-1}. \]

Therefore if \( n \) is odd, then
\[ \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is odd}} \binom{n-1}{n-\ell} = \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is even}} \binom{n-1}{n-\ell} = 2^{n-2} - 1, \quad \text{and} \]
\[ \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is even}} \binom{n-1}{n-\ell} = \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is odd}} \binom{n-1}{n-\ell} = 2^{n-2}. \]

Similarly, if \( n \) is even, then
\[ \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is odd}} \binom{n-1}{n-\ell} = 2^{n-2} \quad \text{and} \quad \sum_{1 \leq \ell \leq n-1 \atop \ell \text{ is even}} \binom{n-1}{n-\ell} = 2^{n-2} - 1. \]

Hence \( -2^{n-2} \leq g(x_1, x_2, \ldots, x_n) \leq 2^{n-2} - 1 \), as required. Next we show that the lower bound \( -2^{n-2} \) and the upper bound \( 2^{n-2} - 1 \) are actually the minimum and the maximum of \( g(x_1, x_2, \ldots, x_n) \), respectively. Recall that the fractional part of a real number \( x \), denoted by \( \{x\} \), is defined by \( \{x\} = x - \lfloor x \rfloor \). Let \( x_k = \frac{1}{2} \) for every \( k = 1, 2, \ldots, n \). Then
\[
g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq k \leq n} (-1)^{k-1} \left\lfloor \frac{k}{2} \right\rfloor \binom{n}{k} \\
= \sum_{1 \leq k \leq n} (-1)^{k-1} \left( \frac{k}{2} \right) \binom{n}{k} - \sum_{1 \leq k \leq n} (-1)^{k-1} \left\{ \frac{k}{2} \right\} \binom{n}{k} \\
= \frac{1}{2} \sum_{1 \leq k \leq n} (-1)^{k-1} k \binom{n}{k} - \frac{1}{2} \sum_{1 \leq k \leq n} \binom{n}{k}, \quad (19)\]

where the last equality is obtained from the fact that \( \left\{ \frac{k}{2} \right\} = 0 \) if \( k \) is even and \( \left\{ \frac{k}{2} \right\} = \frac{1}{2} \) if \( k \) is odd. By differentiating both sides of
\[ (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad (20)\]
and substituting \( x = -1 \), we obtain a well-known identity
\[ \sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} = 0, \quad \text{which holds for all } n \geq 2. \quad (21)\]
In addition, we know that
\[ \sum_{1 \leq k \leq n \text{ is odd}} \binom{n}{k} = 2^{n-1}. \]

Therefore (19) becomes
\[ g(x_1, x_2, \ldots, x_n) = 0 - \frac{1}{2} \left(2^{n-1}\right) = -2^{n-2}. \]

This shows that \(-2^{n-2}\) is the minimum value of \(g\). Next let \(x_k = \frac{1}{2} - \frac{1}{n^2}\) for every \(k = 1, 2, \ldots, n\). Then
\[ g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq k \leq n} (-1)^{k-1} \left\lfloor \frac{k - \frac{1}{n^2}}{n^2} \right\rfloor \binom{n}{k}. \quad (22) \]

If \(1 \leq k \leq n\) and \(k\) is even, then \(\left\lfloor \frac{k - \frac{1}{n^2}}{n^2} \right\rfloor = \frac{k}{2} - 1 = \left\lfloor \frac{k-1}{2} \right\rfloor\). If \(1 \leq k \leq n\) and \(k\) is odd, then \(\left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k}{2} + \frac{1}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor\). Therefore (22) becomes
\[ g(x_1, x_2, \ldots, x_n) = \sum_{1 \leq k \leq n} (-1)^{k-1} \left\lfloor \frac{k - 1}{2} \right\rfloor \binom{n}{k}. \quad (23) \]

Now we can evaluate the sum (23) by using the same method as in (19). We write \(\left\lfloor \frac{k-1}{2} \right\rfloor = \frac{k-1}{2} - \left\{ \frac{k-1}{2} \right\}\) and we know that \(\left\{ \frac{k-1}{2} \right\} = 0\) if \(k\) is odd and \(\left\{ \frac{k-1}{2} \right\} = \frac{1}{2}\) if \(k\) is even. Then (23) can be written as
\[ g(x_1, x_2, \ldots, x_n) = \frac{1}{2} \sum_{1 \leq k \leq n} (-1)^{k-1} \binom{n}{k} - \frac{1}{2} \sum_{1 \leq k \leq n} (-1)^{k-1} \binom{n}{k} + \frac{1}{2} \sum_{1 \leq k \leq n \text{ is even}} \binom{n}{k}. \]

The first sum is zero by (21). The second sum is 1 by substituting \(x = -1\) in (20). Therefore
\[ g(x_1, x_2, \ldots, x_n) = 0 - \frac{1}{2} + \frac{1}{2} \left(2^{n-1} - 1\right) = 2^{n-2} - 1. \]

Recall that the Binet forms of Jacobsthal numbers \(J_n\) and Jacobsthal-Lucas numbers \(j_n\) are
\[ J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n \quad (24) \]
for every \(n \geq 0\). Therefore we obtain the connection between Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal [4] and Tverberg [6] as follows.

**Corollary 5.** If \(n\) is odd, then the maximum and the minimum value of \(g(x_1, x_2, x_3, \ldots, x_n)\) are \(j_{n-2}\) and \(-1 - j_{n-2}\), respectively. If \(n\) is even, then the maximum and the minimum value of \(g(x_1, x_2, x_3, \ldots, x_n)\) are \(3J_{n-2}\) and \(1 - j_{n-2}\), respectively.
Proof. This follows immediately from (24) and Theorem 4.

\[\square\]

Remark 6. From this point on, we will apply the well-known identities which are already recalled without reference.

Next we give the extreme values of \(f_{a_1, a_2, \ldots, a_\ell; m}(k)\). Although we can write \(f_{a_1, a_2, \ldots, a_\ell; m}(k)\) in terms of \(g(x_1, x_2, \ldots, x_n)\) as given in Proposition 3, we do not know the proof which applies Theorem 4 to obtain Theorem 8. Nevertheless, we can use the same idea in the proof of Theorem 4 together with the following lemma to prove Theorem 8.

**Lemma 7.** The following statements hold.

(i) For each \(i \in \{1, 2, \ldots, n\}\) and \(q \in \mathbb{Z}\), we have

\[g(x_1, x_2, \ldots, x_i + q, \ldots, x_n) = g(x_1, x_2, \ldots, x_n).\]

In particular, \(g\) has period 1 in each variable.

(ii) For each \(i \in \{1, 2, \ldots, \ell\}\) and \(q \in \mathbb{Z}\), we have

\[f_{a_1, a_2, \ldots, a_i + q m, \ldots, a_\ell; m}(k) = f_{a_1, a_2, \ldots, a_\ell; m}(k + q m).\]

In particular, \(f\) has period \(m\) in each variable \(a_1, a_2, \ldots, a_\ell\) and \(k\).

Proof. Since \([q + x] = q + [x]\) for every \(q \in \mathbb{Z}\) and \(x \in \mathbb{R}\), we see that

\[
g(x_1, x_2, \ldots, x_i + q, \ldots, x_n) = \left( q + \sum_{i=1}^{n} [x_i] \right) - \left( \binom{n-1}{1} q + \sum_{1 \leq i_1 < i_2 \leq n} [x_{i_1} + x_{i_2}] \right) + \left( \binom{n-1}{2} q + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} [x_{i_1} + x_{i_2} + x_{i_3}] \right) - \cdots + (-1)^{n-1} \left( \binom{n-1}{n-1} q + [x_1 + x_2 + \cdots + x_n] \right) = g(x_1, x_2, \ldots, x_n) + q \sum_{0 \leq k \leq n-1} (-1)^k \binom{n-1}{k}
\]

This proves (i). Next we prove (ii). By Proposition 3 and by (i), we obtain

\[f_{a_1, a_2, \ldots, a_i + q m, \ldots, a_\ell; m}(k) = (-1)^\ell g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_i}{m} + q, \ldots, \frac{a_\ell}{m} \right) + (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_i}{m} + q, \ldots, \frac{a_\ell}{m} \right) = (-1)^\ell g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m} \right) + (-1)^{\ell-1} g \left( \frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m} \right) = f_{a_1, a_2, \ldots, a_\ell; m}(k).
\]

Similarly, \(f_{a_1, a_2, \ldots, a_\ell; m}(k + q m) = f_{a_1, a_2, \ldots, a_\ell; m}(k)\). This completes the proof. \[\square\]
Theorem 8. For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell, k \in \mathbb{Z}$ and $m \geq 1$, we have

$$-2^{\ell-2} \leq f_{a_1,a_2,\ldots,a_\ell;m}(k) \leq 2^{\ell-2}.$$ 

Moreover, $-2^{\ell-2}$ and $2^{\ell-2}$ are best possible in the sense that there are $a_1, a_2, \ldots, a_\ell, m, k$ which make the inequality becomes equality. More precisely the following statements hold.

(i) If $\ell$ is odd, $m$ is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = -2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = 2^{\ell-2}$.

(ii) If $\ell$ is even, $m$ is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2}$.

Proof. By Lemma 7, we can assume that $a_i \in [0, m - 1]$ for every $1 \leq i \leq \ell$. Therefore

$$\left\lfloor \frac{a_i}{m} \right\rfloor = 0 \text{ for every } i \in \{1, 2, \ldots, \ell\}. \quad (25)$$

If $\ell = 2$, then the result follows from (25) and (3), and we have

$$0 \leq \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 1, \quad (26)$$

and

$$-1 \leq - \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \leq 0. \quad (27)$$

Summing (26) and (27), we obtain $-1 \leq f_{a_1,a_2;m}(k) \leq 1$. The result when $\ell \geq 3$ is based on a careful selection of pairs and the case $\ell = 2$. For illustration purpose, we first give a proof for the case $\ell = 3$ and $\ell = 4$. Recall that

$$f_{a_1,a_2,a_3;m}(k) = \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor$$

$$+ \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor.$$

We obtain by (3) and (25) that

$$0 \leq \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \leq 1, \quad (28)$$

$$-1 \leq - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 0, \quad (29)$$

$$-1 \leq - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor \leq 0, \quad (30)$$
\[ 0 \leq \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \leq 1. \]  

(31)

Summing (28), (29), (30), and (31), we see that the middle term is \( f_{a_1, a_2, a_3, m}(k) \). Therefore \(-2 \leq f_{a_1, a_2, a_3, m}(k) \leq 2\). Next we consider

\[
f_{a_1, a_2, a_3, a_4, m}(k) = \left[ \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right] - \left[ \frac{a_1 + a_2 + a_3 + k}{m} \right] - \left[ \frac{a_1 + a_2 + a_4 + k}{m} \right]
- \left[ \frac{a_1 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_1 + a_3 + k}{m} \right] + \left[ \frac{a_2 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_2 + a_3 + k}{m} \right] + \left[ \frac{a_2 + a_4 + k}{m} \right] + \left[ \frac{a_3 + a_4 + k}{m} \right] - \left[ \frac{a_1 + k}{m} \right] - \left[ \frac{a_2 + k}{m} \right] - \left[ \frac{a_3 + k}{m} \right] - \left[ \frac{a_4 + k}{m} \right] + \left[ \frac{k}{m} \right].
\]

Again, we obtain by (3) and (25) the following inequalities:

(32)

(33)

(34)

(35)

(36)

(37)

(38)

(39)

Summing (32) to (39), we see that \(-4 \leq f_{a_1, a_2, a_3, a_4, m}(k) \leq 4\).

Next we prove the general case \( \ell \geq 5 \). The expression of the form \( \left[ \frac{a_{i_1} + a_{i_2} + \ldots + a_{i_r} + k}{m} \right] \) will be called an \( r \)-bracket. So for each \( 1 \leq r \leq \ell \), there are \( \binom{\ell}{r} \) \( r \)-brackets appearing in the sum.
defining \( f_{a_1, a_2, \ldots, a_\ell; m}(k) \). We follow closely the method used in the proof of Theorem 4. So we first pair up the \( \ell \)-bracket with an \((\ell - 1)\)-bracket as follows:

\[
s_1 = \left\lfloor \frac{a_1 + a_2 + \cdots + a_\ell + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + \cdots + a_{\ell-1} + k}{m} \right\rfloor, \tag{40}
\]

and we can apply (3) and (25) to obtain the bound for \( s_1 \). Next we pair up the remaining \((\ell - 1)\)-brackets with \((\ell - 2)\)-brackets as follows:

\[
- \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-1}} + k}{m} \right\rfloor \left. + \right. \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right\rfloor, \tag{41}
\]

and we sum (41) over all \( 1 \leq i_1 < i_2 < \cdots < i_{\ell-1} \leq \ell \) and call it \( s_2 \). Since \( a_\ell \) does not appear in the second term on the right hand side of (40), the term \( a_{i_{\ell-1}} \) appearing in (41) is always \( a_\ell \). So in fact

\[
s_2 = - \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-2} < \ell} \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + a_\ell + k}{m} \right\rfloor + \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-2} < \ell} \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right\rfloor.
\]

We continue doing this process as follows. For each \( 1 \leq r \leq \ell \), let \( c_r \) be the sum of all \( \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r} + k}{m} \right\rfloor \) with \( 1 \leq i_1 < i_2 < \cdots < i_r \leq \ell \), \( a_r \) the sum of all such terms with \( i_r = \ell \), and \( b_r \) the sum of all such terms with \( i_r < \ell \). Therefore \( c_r = a_r + b_r \), the number of summands of \( c_r \) is \( \binom{\ell}{r} \), the number of summands of \( a_r \) is \( \binom{\ell - 1}{r - 1} \), and the number of summands of \( b_r \) is \( \binom{\ell - 1}{r} \). As usual, the empty sum is defined to be zero, so \( b_\ell = 0 \). We have \( s_1 = a_\ell - b_{\ell-1} \) and \( s_2 = -a_{\ell-1} + b_{\ell-2} \). In general, for each \( 1 \leq r \leq \ell - 1 \), we let

\[
s_r = (-1)^{r+1} a_{\ell-r+1} + (-1)^r b_{\ell-r} \quad \text{and} \quad s_\ell = (-1)^{\ell+1} a_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor.
\]

Then

\[
0 \leq s_r \leq \binom{\ell - 1}{\ell - r} \quad \text{if } r \text{ is odd, and} \quad -\binom{\ell - 1}{\ell - r} \leq s_r \leq 0 \quad \text{if } r \text{ is even},
\]
\[
\sum_{1 \leq r \leq \ell} s_r = a_\ell + \sum_{2 \leq r \leq \ell - 1} (-1)^{r+1}a_{\ell-r+1} + \sum_{1 \leq r \leq \ell - 2} (-1)^r b_{\ell-r} + (-1)^{\ell-1}b_1 + s_\ell
\]
\[
= a_\ell + \sum_{1 \leq r \leq \ell - 2} (-1)^r(a_{\ell-r} + b_{\ell-r}) + (-1)^{\ell-1}b_1 + (-1)^{\ell+1}a_1 + \left\lfloor \frac{k}{m} \right\rfloor
\]
\[
= c_\ell + \sum_{1 \leq r \leq \ell - 2} (-1)^rc_{\ell-r} + (-1)^{\ell-1}c_1 + \left\lfloor \frac{k}{m} \right\rfloor
\]
\[
= \sum_{0 \leq r \leq \ell - 1} (-1)^rc_{\ell-r} + \left\lfloor \frac{k}{m} \right\rfloor
\]
\[
= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\]

Therefore
\[
- \sum_{1 \leq r \leq \ell \atop r \text{ is even}} \binom{\ell - 1}{\ell - r} \leq f_{a_1,a_2,\ldots,a_\ell;m}(k) \leq \sum_{1 \leq r \leq \ell \atop r \text{ is odd}} \binom{\ell - 1}{\ell - r}.
\]

Replacing \( r \) by \( r + 1 \), we see that
\[
\sum_{1 \leq r \leq \ell \atop r \text{ is odd}} \binom{\ell - 1}{\ell - r} = \sum_{0 \leq r \leq \ell - 1 \atop r \text{ is even}} \binom{\ell - 1}{\ell - 1 - r} = 2^{\ell-2}.
\]

Similarly,
\[
- \sum_{1 \leq r \leq \ell \atop r \text{ is even}} \binom{\ell - 1}{\ell - r} = -2^{\ell-2}.
\]

Therefore
\[
-2^{\ell-2} \leq f_{a_1,a_2,\ldots,a_\ell;m}(k) \leq 2^{\ell-2}, \tag{42}
\]
as required. If \( \ell \) is odd, \( m \) is even, and \( a_i = \frac{m}{2} \) for every \( 1 \leq i \leq \ell \), we obtain by Proposition 3 and Theorem 4 that \( f_{a_1,a_2,\ldots,a_\ell;m}(0) = g(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = -2^{\ell-2} \) and \( f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = (-1)^\ell g(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) + (-1)^{\ell-1}g(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = 2^{\ell-2} \). If \( \ell \) is even, \( m \) is even, and \( a_i = \frac{m}{2} \) for every \( 1 \leq i \leq \ell \), we obtain similarly that \( f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2} \) and \( f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2} \). So \( 2^{\ell-2} \) and \( -2^{\ell-2} \) in (42) cannot be improved. This completes the proof.

We obtain the extreme values of \( S_{a_1,a_2,\ldots,a_\ell;m} \) for some cases \( \ell \geq 4 \) as well. More precisely, we have the following result.

**Theorem 9.** For each \( \ell \geq 2 \), \( a_1, a_2, \ldots, a_\ell \in \mathbb{Z} \), \( m \in \mathbb{N} \), and \( K \in \mathbb{N} \cup \{0\} \), we have
\[
-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor \leq S_{a_1,a_2,\ldots,a_\ell;m}(K) \leq 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor. \tag{43}
\]
Moreover, if \( \ell \) is odd, then the lower bound \(-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor\) is sharp and if \( \ell \) is even, then the upper bound \(2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor\) is sharp in the sense that there are \(a_1, a_2, \ldots, a_\ell, m, k\) which make the inequality become equality. More precisely, the following statements hold.

(i) If \( \ell \) is odd, \( m \) is even, and \( a_i = \frac{m}{2} \) for every \( i = 1, 2, \ldots, \ell \), then \( S_{a_1, a_2, \ldots, a_\ell; m}(K) = -2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor\).

(ii) If \( \ell \) is even, \( m \) is even, and \( a_i = \frac{m}{2} \) for every \( i = 1, 2, \ldots, \ell \), then \( S_{a_1, a_2, \ldots, a_\ell; m}(K) = 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor\).

Proof. If \( \ell = 2 \), then the result is already proved by Jacobsthal [4]. See also another proof by Tverberg [6]. We recall the result when \( \ell = 2 \) for easy reference as follows:

\[ 0 \le S_{a,b;m}(K) \le \left\lfloor \frac{m}{2} \right\rfloor. \quad (44) \]

As before the result when \( \ell \ge 3 \) is based on the case \( \ell = 2 \) and a careful selection of pairs, and we first illustrate the idea by giving the proof for the case \( \ell = 3 \) and \( \ell = 4 \). Recall that

\[
f_{a_1, a_2, a_3; m}(k) = \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor.
\]

We have

\[
f_{a_1 + a_2, a_3; m}(k) = \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_3 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor,
\]

\[
-f_{a_1, a_3; m}(k) = - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor,
\]

\[
-f_{a_2, a_3; m}(k) = - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor.
\]

Summing (45), (46), and (47), we see that

\[
f_{a_1, a_2, a_3; m}(k) = f_{a_1 + a_2, a_3; m}(k) - f_{a_1, a_3; m}(k) - f_{a_2, a_3; m}(k).
\]

By the definition of \( S_{a_1, a_2, a_3; m}(K) \), (48), and (44), we obtain

\[
S_{a_1, a_2, a_3; m}(K) = \sum_{k=0}^{K} f_{a_1, a_2, a_3; m}(k)
\]

\[
= \sum_{k=0}^{K} f_{a_1 + a_2, a_3; m}(k) - \sum_{k=0}^{K} f_{a_1, a_3; m}(k) - \sum_{k=0}^{K} f_{a_2, a_3; m}(k)
\]

\[
= S_{a_1 + a_2, a_3; m}(K) - S_{a_1, a_3; m}(K) - S_{a_2, a_3; m}(K)
\]

\[
\ge 0 - \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor = -2 \left\lfloor \frac{m}{2} \right\rfloor.
\]
Similarly,
\[ S_{a_1, a_2, a_3: m}(K) \leq \left\lfloor \frac{m}{2} \right\rfloor - 0 - 0 = \left\lfloor \frac{m}{2} \right\rfloor \leq 2 \left\lfloor \frac{m}{2} \right\rfloor. \]

Similarly, we have the following equalities:
\[
\begin{align*}
-f_{a_1+a_2+a_3, a_4; m}(k) &= -\left[ \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right] - \left[ \frac{a_1 + a_2 + a_4 + k}{m} \right] - \left[ \frac{a_4 + k}{m} \right] + \left[ \frac{k}{m} \right], \\
f_{a_1+a_2+a_3, a_4; m}(k) &= \left[ \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_1 + a_2 + k}{m} \right] + \left[ \frac{a_4 + k}{m} \right] - \left[ \frac{k}{m} \right], \\
f_{a_2+a_3, a_4; m}(k) &= \left[ \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_2 + a_3 + k}{m} \right] + \left[ \frac{a_4 + k}{m} \right] - \left[ \frac{k}{m} \right], \\
-f_{a_2+a_3+a_4, a_4; m}(k) &= -\left[ \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_2 + a_3 + a_4 + k}{m} \right] + \left[ \frac{a_4 + k}{m} \right] - \left[ \frac{k}{m} \right].
\end{align*}
\]

Summing (49) to (55) and recalling the definition of \( f_{a_1, a_2, a_3, a_4: m}(k) \), we see that
\[
f_{a_1, a_2, a_3, a_4: m}(k) = f_{a_1+a_2+a_3, a_4; m}(k) - f_{a_1+a_2, a_3, a_4; m}(k) - f_{a_1+a_3, a_4; m}(k) - f_{a_2+a_3, a_4; m}(k) - f_{a_2+a_3+a_4, a_4; m}(k)
\]
\[+ f_{a_1, a_4; m}(k) + f_{a_2, a_4; m}(k) + f_{a_3, a_4; m}(k). \]  

(56)

Then we obtain from (56) and (44) that
\[
\begin{align*}
S_{a_1, a_2, a_3, a_4: m}(K) &= S_{a_1+a_2+a_3, a_4: m}(K) - S_{a_1+a_2, a_3, a_4; m}(K) - S_{a_1+a_3, a_4; m}(K) - S_{a_2+a_3, a_4; m}(K) \\
&\quad + S_{a_1, a_4; m}(K) + S_{a_2, a_4; m}(K) + S_{a_3, a_4; m}(K) \\
&\leq \left\lfloor \frac{m}{2} \right\rfloor - 0 - 0 + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor = 4 \left\lfloor \frac{m}{2} \right\rfloor.
\end{align*}
\]

Similarly, \( S_{a_1, a_2, a_3, a_4: m}(K) \geq -4 \left\lfloor \frac{m}{2} \right\rfloor \). Next we prove the general case \( \ell \geq 5 \). The expression of the form \( \left\lfloor \frac{a_1 + a_2 + \cdots + a_r + k}{m} \right\rfloor \) will be called an \( r \)-bracket. So for each \( 0 \leq r \leq \ell \), there are \( \binom{\ell}{r} \) \( r \)-brackets appearing in the sum defining \( f_{a_1, a_2, \ldots, a_\ell: m}(k) \). We first pair up the \( \ell \)-bracket with an \( (\ell - 1) \)-bracket, a 1-bracket and a 0-bracket as follows:
\[
s_1(k) = \left[ \frac{a_1 + a_2 + \cdots + a_\ell + k}{m} \right] - \left[ \frac{a_1 + a_2 + \cdots + a_{\ell-1} + k}{m} \right] - \left[ \frac{a_{\ell} + k}{m} \right] + \left[ \frac{k}{m} \right].
\]  

(57)
So $s_1(k)$ is in fact $f_{a_1 + a_2 + \cdots + a_{\ell-1}, a_\ell; m}(k)$ and we can apply (44) to obtain the inequality

$$0 \leq S_{a_1 + a_2 + \cdots + a_{\ell-1}, a_\ell; m}(K) = \sum_{k=0}^{K} s_1(k) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$  

Next we pair up the remaining $(\ell-1)$-brackets with $(\ell-2)$-brackets, 1-brackets and 0-brackets as follows:

$$-\left[ \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-1}} + k}{m} \right] + \left[ \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right] + \left[ \frac{a_{i_{\ell-1}} + k}{m} \right] - \left\lfloor \frac{k}{m} \right\rfloor, \quad (58)$$

and we sum (58) over all $1 \leq i_1 < i_2 < \cdots < i_{\ell-1} \leq \ell$ and call it $s_2(k)$. Since $a_\ell$ does not appear in the second term on the right hand side of (57), the term $a_{i_{\ell-1}}$ appearing in (58) is always $a_\ell$. So in fact (58) is $-f_{a_1 + a_2 + \cdots + a_{\ell-2}, a_\ell; m}(k)$ and

$$s_2(k) = -\sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-2} < \ell} f_{a_1 + a_2 + \cdots + a_{i_{\ell-2}}, a_\ell; m}(k)$$

Furthermore,

$$\sum_{k=0}^{K} s_2(k) = -\sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-2} < \ell} S_{a_1 + a_2 + \cdots + a_{i_{\ell-2}}, a_\ell; m}(K) \leq 0,$$

where the last inequality is obtained from (44). We continue doing this process and follow closely the method used in the proof of Theorems 4 and 8. The well-known identities previously recalled will be applied without reference. For each $1 \leq r \leq \ell$, let $c_r(k)$ be the sum of all $\left[ \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r} + k}{m} \right]$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq \ell$, $a_r(k)$ the sum of all such terms with $i_r = \ell$, and $b_r(k)$ the sum of all such terms with $i_r < \ell$. Therefore $c_r(k) = a_r(k) + b_r(k)$, the number of $r$-brackets appearing in the sum defining $c_r(k)$ is $\binom{\ell}{r}$, the number of $r$-brackets appearing in the sum defining $a_r(k)$ is $\binom{\ell-1}{r-1}$, and the number of $r$-brackets appearing in the sum defining $b_r(k)$ is $\binom{\ell-1}{r}$. As usual, the empty sum is defined to be zero, so $b_0(k) = 0$. We have $s_1(k) = a_\ell(k) - b_{\ell-1}(k) - a_{i_1}(k) + \left\lfloor \frac{k}{m} \right\rfloor$ and $s_2(k) = -a_{\ell-1}(k) + b_{\ell-2}(k) + \binom{\ell-1}{\ell-2} a_1(k) - \binom{\ell-1}{\ell-2} \left\lfloor \frac{k}{m} \right\rfloor$. In general, for each $1 \leq r \leq \ell-1$, we let

$$s_r(k) = (1)^{r+1} a_{\ell-r+1}(k) + (-1)^r b_{\ell-r}(k) + (-1)^r \binom{\ell-1}{\ell-r} a_1(k) + (-1)^{r+1} \binom{\ell-1}{\ell-r} \left\lfloor \frac{k}{m} \right\rfloor$$

$$= (1)^{r+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-r} < \ell} f_{a_1 + a_2 + \cdots + a_{i_{\ell-r}}, a_\ell; m}(k).$$

Then

$$\sum_{k=0}^{K} s_r(k) = (1)^{r+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-r} < \ell} S_{a_1 + a_2 + \cdots + a_{i_{\ell-r}}, a_\ell; m}(K).$$
So by (44), we see that

\[
0 \leq \sum_{k=0}^{K} s_r(k) \leq \left( \ell - 1 \right) \left( \frac{m}{2} \right) \text{ if } r \text{ is odd, and } - \left( \ell - 1 \right) \left( \frac{m}{2} \right) \leq \sum_{k=0}^{K} s_r(k) \leq 0 \text{ if } r \text{ is even.}
\]

Similar to the proof of Theorems 4 and 8, we obtain

\[
\sum_{1 \leq r \leq \ell - 1} s_r(k) = a_\ell + \sum_{2 \leq r \leq \ell - 1} (-1)^{r+1} a_{\ell-r+1} + \sum_{1 \leq r \leq \ell - 2} (-1)^r b_{\ell-r} + (-1)^\ell b_1
\]

\[
+ (-1)^{\ell+1} a_1 + (-1)^\ell \left[ \frac{k}{m} \right]
\]

\[
= a_\ell + \sum_{1 \leq r \leq \ell - 2} (-1)^r (a_{\ell-r} + b_{\ell-r}) + (-1)^\ell b_1 + (-1)^{\ell-1} a_1 + (-1)^\ell \left[ \frac{k}{m} \right]
\]

\[
= c_\ell + \sum_{1 \leq r \leq \ell - 2} (-1)^r c_{\ell-r} + (-1)^\ell c_1 + (-1)^\ell \left[ \frac{k}{m} \right]
\]

\[
= \sum_{0 \leq r \leq \ell - 1} (-1)^r c_{\ell-r} + (-1)^\ell \left[ \frac{k}{m} \right]
\]

\[
= f_{a_1, a_2, \ldots, a_\ell; m}(k).
\]

Therefore

\[
- \sum_{1 \leq r \leq \ell - 1} \left( \ell - 1 \right) \left( \frac{m}{2} \right) \leq \sum_{k=0}^{K} f_{a_1, a_2, \ldots, a_\ell; m}(k) \leq \sum_{1 \leq r \leq \ell - 1} \left( \ell - 1 \right) \left( \frac{m}{2} \right).
\]  

(59)

The middle term in (59) is $S_{a_1, a_2, \ldots, a_\ell; m}(K)$. The left and right most terms in (59) are, respectively, equal to $-2^{\ell-2} \left[ \frac{m}{2} \right] \text{ and } 2^{\ell-2} \left[ \frac{m}{2} \right]$, which can be evaluated by the well-known identity previously recalled. This proves the first part of the theorem. Next we show that one of the upper bound or lower bound is sharp. Let $C = \{a_1, a_2, \ldots, a_\ell\}$. Suppose $\ell$ is odd, $m$ is even, and $a_i = \frac{m}{2}$ for every $1 \leq i \leq \ell$. Then we obtain by Proposition 3 and Theorem 4 that $f_{C; m}(0) = g(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = -2^{\ell-2}$. Let $0 < k < \frac{m}{2}$. By the definition of $f_{C; m}(k)$, we see that

\[
f_{C; m}(k) = \sum_{T \subseteq [1, \ell]} (-1)^{\ell-|T|} \left( \frac{k}{m} + \left\lfloor \frac{|T|}{2} \right\rfloor \right)
\]

\[
= \sum_{r=0}^{\ell} (-1)^{\ell-r} \left( \frac{\ell}{r} \right) \left( \frac{k}{m} + \left\lfloor \frac{r}{2} \right\rfloor \right)
\]

(60)

Since $0 < k < \frac{m}{2}$, we have $\frac{r}{2} < \frac{k}{m} + \frac{r}{2} < \frac{k + 1}{2}$. So if $r$ is even, then $\left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor = \frac{r}{2} = \left\lfloor \frac{r}{2} \right\rfloor$. In any case, $\left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor = \frac{r}{2} = \left\lfloor \frac{a}{m} + \frac{r}{2} \right\rfloor$. This implies
that $f_{C;m}(k) = f_{C;m}(0)$ for every $k = 0, 1, 2, \ldots, \frac{m}{2} - 1$. Then

$$S_{C;m}\left(\frac{m}{2} - 1\right) = \sum_{k=0}^{\frac{m}{2}-1} f_{C;m}(k) = \frac{m}{2} f_{C;m}(0) = -2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$$

So $-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ in (43) cannot be improved when $\ell$ is odd. Next suppose $\ell$ is even, $m$ is even, and $a_i = \frac{m}{2}$ for every $1 \leq i \leq \ell$. Then we obtain similarly that $f_{C;m}(k) = f_{C;m}(0) = 2^{\ell-2}$ for every $k = 0, 1, 2, \ldots, \frac{m}{2} - 1$. Then $S_{C;m}(\frac{m}{2} - 1) = 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$. So $2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ in (43) cannot be improved when $\ell$ is even. This completes the proof.

\[\Box\]

3 Acknowledgments

Kritkhajohn Onphaeng wishes to thank DPST (Development and Promotion of Science and Technology Talents) for giving him a scholarship. Prapanpong Pongsriiam receives financial support jointly from The Thailand Research Fund and Faculty of Science, Silpakorn University, grant number RSA5980040. Correspondence should be addressed to Prapanpong Pongsriiam: prapanpong@gmail.com.

References


2010 Mathematics Subject Classification: Primary 11A25; Secondary 11B37.

Keywords: Jacobsthal sum, Jacobsthal number, Jacobsthal-Lucas number, floor function, sum.