The $r$th Moment of the Divisor Function: 
An Elementary Approach

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Abstract
For integer $r \geq 1$ we give an elementary proof for the main term of the asymptotic behavior of the $r$th moment of the number of divisors of $n$ for positive integers $n \leq x$. 

1 Introduction

Let $\tau(n)$ be the number of divisors of $n$. Ramanujan [2] stated without proof that, given any real number $\varepsilon > 0$, the estimate

$$\sum_{n \leq x} \tau(n)^2 = x(A(\log x)^3 + B(\log x)^2 + C\log x + D) + O(x^{3/5+\varepsilon})$$

holds with $A = \pi^2 - 2$. An elementary proof of the asymptotic formula

$$\sum_{n \leq x} \tau(n)^2 \sim Ax(\log x)^3,$$

as $x \to \infty$, appears in several places (see, for example, [1, Thm. 7.8]). Wilson [3] proved Ramanujan’s claim and generalized it by showing that for any integer $r \geq 2$ one has

$$\sum_{n \leq x} \tau(n)^r = x(C_{r,1}(\log x)^{2r-1} + C_{r,2}(\log x)^{2r-2} + \cdots + C_{r,2^r}) + O(x^{2r^2+\varepsilon}).$$

Note that when $r = 2$, Wilson’s error term is better than the one claimed by Ramanujan. We are not aware even of elementary proofs for the asymptotic formula

$$\sum_{n \leq x} \tau(n)^r \sim C_rx(\log x)^{2r-1}$$

as $x \to \infty$ for any $r \geq 2$. In this note, we give an elementary proof of the following more general result.

**Theorem 1.** Let $k$ be a positive integer and $f(n)$ be a multiplicative function which on prime powers $p^\alpha$ satisfies

$$f(p) = k \quad \text{and} \quad f(p^\alpha) = \alpha \cdot O(1) \quad \text{for all primes } p \text{ and integers } \alpha \geq 2,$$

where the constant implied by the above $O$ is uniform in $p$. Then

$$\sum_{n \leq x} f(n) = xC_f(\log x)^{k-1} + O(x(\log x)^{k-2})$$

where

$$C_f = \frac{1}{(k-1)!} \left( \prod_{p \geq 2} \left( 1 - \frac{1}{p} \right)^k \left( \sum_{\alpha \geq 0} \frac{f(p^n)}{p^n} \right) \right).$$

In the case $f(n) = \tau(n)^r$ for integer $r \geq 1$, Theorem 1 applies with $k = 2^r$.

The only facts that we use are Abel’s summation formula, the Möbius inversion formula, the elementary estimate

$$\sum_{n \leq t} \frac{1}{n} = \log t + \gamma + O(1/t) \quad (1)$$

valid for all real $t \geq 1$, and the fact that the counting function of the squarefull numbers $s \leq t$ is $O(t^{1/2})$, where $s$ is squarefull if and only if $p^2 \mid s$ for all prime factors $p$ of $s$, all provable by elementary means.
2 A lemma

Lemma 2. Assume that \( r \) is a positive integer and \( f(n) \) is some arithmetic function such that

\[
\sum_{n \leq x} f(n) = \sum_{j=0}^{r} c_j (\log x)^j + O(x^{-1/2+o(1)}),
\]

(2)

for some constants \( c_j \), \( j = 0, \ldots, r \). Then

\[
\sum_{n \leq x} f(n)(\log(x/n))^k = \sum_{\ell=0}^{k+r} C_\ell (\log x)^\ell + O(x^{-1/2+o(1)}),
\]

(3)

holds for all positive integers \( k \) with some constants \( C_0, \ldots, C_{k+r} \). Here, if \( \ell \in \{k, k+1, \ldots, k+r\} \), then

\[
C_\ell := c_{\ell-k} \left( 1 + (\ell - k) \sum_{i=1}^{k} \frac{(-1)^i}{\ell - k + i \left( \begin{array}{c} k \\ i \end{array} \right)} \right).
\]

(4)

Furthermore, if \( r \geq t \geq 1 \) are positive integers and

\[
\sum_{n \leq x} f(n) = \sum_{j=t}^{r} c_j (\log x)^j + O((\log x)^{t-1}),
\]

(5)

then

\[
\sum_{n \leq x} f(n)(\log(x/n))^k = \sum_{j=k+t}^{k+r} C_j (\log x)^j + O((\log x)^{t+k-1}).
\]

(6)

Proof. We show how to deduce (3) out of (2) with the leading coefficients given by (4). Let

\[
A(x) = \sum_{n \leq x} f(n).
\]

Then

\[
A(x) = \sum_{j=0}^{r} c_j (\log x)^j + R(x),
\]

where \(|R(x)| = x^{-1/2+o(1)}\) as \( x \to \infty \). Let \( i \geq 1 \). Put

\[
B_i(x) := \sum_{n \leq x} f(n)(\log n)^i.
\]
Then, by the Abel summation formula and by interchanging the order between the summation and the integration, we get

\[ B_i(x) = A(x)(\log x)^i - i \int_1^x A(t) \left( \frac{(\log t)^{i-1}}{t} \right) dt \]

\[ = \sum_{j=0}^r \left( c_j (\log x)^{j+i} - i \int_1^x \left( c_j (\log t)^{j+i-1} \right) dt \right) \]

\[ - i \int_1^x \frac{(\log t)^{i-1}R(t)}{t} dt + R(x)(\log x)^i \]

\[ = \sum_{j=0}^r \left( c_j (\log x)^{j+i} - c_j^i (\log t)^{j+i} \right) + \]

\[ - i \int_1^x \frac{(\log t)^{i-1}R(t)}{t} dt + i \int_x^\infty \frac{(\log t)^{i-1}R(t)}{t} dt + R(x)(\log x)^i \]

\[ = \sum_{j=0}^r \frac{c_j i^j}{j+i} (\log x)^{j+i} + D_i + O(x^{-1/2+o(1)}), \]

where

\[ D_i := -i \int_1^\infty \frac{(\log t)^{i-1}R(t)}{t} dt \]

In the above, we used the fact that \(|R(t)| \leq t^{-1/2+o(1)}\) as \(t \to \infty\) to deduce that the above integral converges and that its tail from \(x\) to infinity as well as the other errors are \(O(x^{-1/2+o(1)})\) as \(x \to \infty\). Using the binomial formula and the above arguments, we have

\[ C_k(x) := \sum_{n \leq x} f(n)(\log(x/n))^k \]

\[ = \sum_{i=0}^k (-1)^i \binom{k}{i} (\log x)^{k-i} \sum_{n \leq x} f(n)(\log n)^i \]

\[ = \sum_{n \leq x} f(n) + \sum_{i=1}^k (-1)^i \binom{k}{i} (\log x)^{k-i} B_i(x) \]

\[ = \sum_{\ell=0}^{k+r} C_\ell (\log x)^\ell + O(x^{-1/2+o(1)}), \]

where \(C_\ell\) are given by formula (4) for \(\ell \geq k\). For \(\ell = 1, \ldots, k - 1\), the coefficient \(C_\ell\) involves the expression \(D_\ell\). The deduction of (6) out of (5) is immediate by similar arguments. \(\Box\)
3 The proof of Theorem 1

Let $f_0(n) := f(n)$. Recursively define $f_j(n)$ such that

$$f_{j-1}(n) = \sum_{d|n} f_j(d), \quad j = 1, 2, \ldots.$$ 

By M"obius inversion,

$$f_j(n) = \sum_{d|n} \mu(d) f_{j-1}(n/d).$$

On primes

$$f_j(p) = f_{j-1}(p) - 1, \quad j = 1, 2, \ldots.$$ 

Since $f_0(p) = k$, we get that $f_j(p) = k - j$. In particular, $f_k(p) = 0$. Further, for $\alpha \geq 2$, we have that

$$f_j(p^\alpha) = f_{j-1}(p^\alpha) - f_{j-1}(p^{\alpha-1}).$$

Since $f_0(p^\alpha) = \alpha^{O(1)}$ it follows that $f_j(p^\alpha) = \alpha^{O(1)}$ for all $j \geq 2$. The constant in $O(1)$ might depend on $j$. Further,

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right) \sum_{\alpha \geq 0} \frac{f_{j-1}(p^\alpha)}{p^\alpha}, \quad j = 1, 2, \ldots,$$

therefore

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha}, \quad j = 0, 1, \ldots$$

Put

$$E_j := \prod_{p \geq 2} \left(\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha}\right) = \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha}\right).$$

Fix $j \geq 1$. Then

$$F_{j-1}(x) := \sum_{n \leq x} \frac{f_{j-1}(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d|n} f_j(d) = \sum_{d \leq x} f_j(d) \sum_{\alpha \leq \log x} \frac{1}{n}.$$ 

In the inner sum, we write an $n \leq x$ which is a multiple of $d$ as $n = dm$ for some integer $m \leq x$. We get

$$F_{j-1}(x) = \sum_{d \leq x} \frac{f_j(d)}{d} \sum_{m \leq x/d} \frac{1}{m} = \sum_{d \leq x} \frac{f_j(d)}{d} \left(\log(x/d) + \gamma + O(d/x)\right)$$

$$= \sum_{d \leq x} \frac{f_j(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_j(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} |f_j(d)|\right) \quad (7)$$
for \( j = 1, 2, \ldots \). When \( j = k \), since \( f_k(p) = 0 \), it follows that \( f_k(d) = 0 \) if \( d \) is not squarefull. Thus, when \( j = k \) in the right-hand side of (7), we have

\[
\sum_{d \leq x} \frac{f_k(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_k(d)}{d} + O \left( \frac{1}{x} \sum_{d \leq x} |f_k(d)| \right).
\]

Note that

\[
\sum_{d \leq x} \frac{f_k(d)}{d} = \sum_{d \geq 1} \frac{f_k(d)}{d} + O \left( \sum_{d > x} \frac{|f_k(d)|}{d} \right) = E_k + O \left( \sum_{d > x} \frac{1}{d^{1 + o(1)}} \right).
\]

(8)

where for the error term we used the fact that \( |f_k(d)| = |\tau(d)|^{O(1)} = d^{o(1)} \) as \( d \to \infty \) and the Abel summation formula to conclude that

\[
\sum_{d > x} \frac{1}{d^{1 + o(1)}} \leq x^{-1/2 + o(1)} \quad \text{as} \quad x \to \infty.
\]

Further, we have

\[
\sum_{d \leq x} \frac{f_k(d)}{d} (-\log d + \gamma) = \sum_{d \geq 1} \frac{f_k(d)(-\log d + \gamma)}{d} + O \left( \sum_{d > x} \frac{|f_k(d)| \log d}{d} \right)
\]

:= \quad F_k + O(x^{-1/2 + o(1)})
\]

(9)

as \( x \to \infty \), by a similar argument since \( |f_k(d)| \log d \leq d^{o(1)} \) as \( d \to \infty \). Finally

\[
\sum_{d \leq x} |f_k(d)| \leq x^{1/2 + o(1)},
\]

(10)

again since \( f_k(d) = 0 \) if \( d \) is not squarefull. Collecting (8), (9) and (10) and putting them into (7) with \( j = k \), we get

\[
F_{k-1}(x) = \sum_{n \leq x} \frac{f_{k-1}(n)}{n} = E_k \log x + F_k + O(x^{-1/2 + o(1)}).
\]

In a similar way,

\[
G_{k-1}(x) := \sum_{n \leq x} \frac{|f_{k-1}(n)|}{n} = E_k' \log x + F_k' + O(x^{-1/2 + o(1)}).
\]
for some (maybe different) constants \( E'_k \) and \( F'_k \). We now apply Lemma 2 in order to find recursively \( F_{k-2}(x), F_{k-3}(x), \ldots, F_0(x) \). We claim, by induction on \( j \), that

\[
F_{k-j}(x) = A_j (\log x)^j + B_j (\log x)^{j-1} + O((\log x)^{j-2})
\]

(11)

for \( j = 2, \ldots, k \). At \( j = 1 \), this is so with \( A_1 = E_k, B_1 = F_k \) and the error term is better, namely \( O(x^{-1/2+o(1)}) \). In order to realize the induction step from \( j = 1 \) to \( j = 2 \), we use the first part of Lemma 1 with \( r = 1 \), whereas for the induction step from \( j \geq 2 \) to \( j + 1 \) we use the second part of Lemma 2 with \( r = j \) and \( t = j - 1 \). Assuming that (11) holds for \( j \geq 1 \), we have, by (7),

\[
F_{k-j-1}(x) = \sum_{d \leq x} \frac{f_{k-j-1}(d)}{d} = \sum_{d \leq x} \frac{f_{k-j}(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_{k-j}(d)}{d} + O\left( \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right).
\]

By Lemma 2, we get that the right hand side is

\[
\frac{A_j}{j+1} (\log x)^{j+1} + \left( \frac{B_j}{j} + \gamma A_j \right) (\log x)^j + O\left( (\log x)^{j-1} + \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right)
\]

:= \left( A_j+1 (\log x)^{j+1} + B_j+1 (\log x)^j + O\left( (\log x)^{j-1} + \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right) \right),
\]

where

\[
A_{j+1} = \frac{A_j}{j+1}, \quad \text{and} \quad B_{j+1} = \gamma A_j + \frac{B_j}{j}.
\]

Thus, we note that \( A_j = E_k/j! \). It remains to deal with the sum in the error term. But the exact same approach applies to \(|f_{k-j}(n)|\). That is \( g_0(n) = |f_{k-j}(n)| \) satisfies the same conditions as our initial \( f_0(n) \) with \( k \) replaced by \( k - j \). Thus,

\[
\sum_{d \leq x} \frac{|f_{k-j}(d)|}{d} = C_j (\log x)^j + D_j (\log x)^{j-1} + O((\log x)^{j-2}),
\]

where for \( j = 1 \), the error term is \( O(x^{-1/2+o(1)}) \) as \( x \to \infty \). By Abel summation, we get that

\[
\sum_{d \leq x} |f_{k-j}(d)| = x (C_j (\log x)^j + D_j (\log x)^{j-1} + O((\log x)^{j-2}))
\]

\[
- \int_1^{x} (C_j (\log t)^j + D_j (\log t)^{j-1} + O((\log t)^{j-2})) dt
\]

\[
= O(x (\log x)^{j-1}),
\]

7
which is sufficient for us. This completes the induction procedure and shows that at \( j = k \) we have
\[
\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{k!} E_k (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}).
\]
Abel summation formula once again gives
\[
\sum_{n \leq x} f(n) = \left( \frac{E_k}{k!} (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}) \right) x
- \int_1^x \left( \frac{E_k}{k!} (\log t)^k + B_k (\log t)^{k-1} + O((\log t)^{k-2}) \right) dt
= \frac{E_k}{(k-1)!} x (\log x)^{k-1} + O(x (\log x)^{k-2}),
\]
which is what we wanted.

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References


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