Extending a Recent Result on Hyper $m$-ary Partition Sequences

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Abstract

A hyper $m$-ary partition of an integer $n$ is defined to be a partition of $n$ where each part is a power of $m$ and each distinct power of $m$ occurs at most $m$ times. Let $h_m(n)$ denote the number of hyper $m$-ary partitions of $n$ and consider the resulting sequence. We show that the hyper $m_1$-ary partition sequence is a subsequence of the hyper $m_2$-ary partition sequence, for $2 \leq m_1 < m_2$.

1 Introduction

In 2004, Courtright and Sellers [2] defined a hyper $m$-ary partition of an integer $n$ to be a partition of $n$ for which each part is a power of $m$ and each power of $m$ occurs at most $m$
times. They denote the number of hyper \( m \)-ary partitions of \( n \) as \( h_m(n) \) and showed that they satisfy the following recurrence relation:

\[
\begin{align*}
    h_m(mq) &= h_m(q) + h_m(q-1), \\
    h_m(mq+s) &= h_m(q), \text{ for } 1 \leq s \leq m-1.
\end{align*}
\] (1) (2)

Several of these hyper \( m \)-ary partition sequences can be found in the On-line Encyclopedia of Integer Sequences [7]. In particular, \( h_2 \) is \texttt{A002487}, \( h_3 \) is \texttt{A054390}, \( h_4 \) is \texttt{A277872}, and \( h_5 \) is \texttt{A277873}.

The sequence \( h_2 \) \texttt{A002487}, the hyperbinary partition sequence, is well known. It is commonly known as the Stern sequence based on Stern’s work [8]. Northshield [5] gives an extensive summary of the many uses and applications of \texttt{A002487}. Calkin and Wilf [1] also studied \( h_2 \), outlining a connection between this sequence and a sequence of fractions they defined and used to give an enumeration of the rationals. Since then, several authors have studied similar restricted binary and \( m \)-ary partition functions; see [3, 4, 6] for additional examples.

In this paper, we will be analyzing hyper \( m \)-ary partitions of \( n \) while also considering the base \( m \) representation of \( n \). Thus, it will be convenient to have clear and distinct notation. In particular, for \( m \geq 2 \), let \((n_r, n_{r-1}, \ldots, n_1, n_0)_m\) be the base \( m \) representation of positive integer \( n \) where \( 0 \leq n_i < m \), \( n_r \neq 0 \), and \( n = \sum_{i=0}^{r} n_i m^i \). Also, for \( 2 \leq m_1 < m_2 \) and \( n = (n_r, n_{r-1}, \ldots, n_1, n_0)_{m_1} \), we define a change of base function, \( F_{m_1, m_2}(n) = (n_r, n_{r-1}, \ldots, n_1, n_0)_{m_2} \).

Next, we write a hyper \( m \)-ary partition of \( n \) as \([x_r, x_{r-1}, \ldots, x_1, x_0]_m \) where \( 0 \leq x_i \leq m \) and \( n = \sum_{i=0}^{r} x_i m^i \). Here, we may allow any of the \( x_i \) to be \( 0 \) so that each hyper \( m \)-ary partition of \( n \) is the same length \( r \) as the base \( m \) representation of \( n \). Furthermore, let \( H_m(n) \) be the set of all distinct hyper \( m \)-ary partitions of \( n \). Observe that \( h_m(n) \) is the cardinality of this set.

Recently, the authors gave an identity relating \( h_2 \) to \( h_3 \) and then generalized this identity to show that \( h_2 \) is a subsequence of \( h_m \) for any \( m \) [4]. This result involved giving a bijection between \( H_2(\ell) \) and \( H_m(k) \), where \( k = F_{2, m}(\ell) \). In this note, the authors will follow a similar process to show that \( h_{m_1} \) is a subsequence of \( h_{m_2} \), for \( 2 \leq m_1 < m_2 \).

## 2 A preliminary example

Consider the integer \( 37 = (1, 1, 0, 1)_3 \) and use the change of base function to find the integer with the same digits in base 4. In particular, \( F_{3,4}(37) = (1, 1, 0, 1)_4 = 81 \). Now consider the hyper 3-ary partitions of 37 and the hyper 4-ary partitions of 81.

\[
\begin{align*}
37 &= 1 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0 \\
    &= 1 \cdot 3^3 + 3 \cdot 3^1 + 1 \cdot 3^0 \\
    &= 3 \cdot 3^2 + 3 \cdot 3^1 + 1 \cdot 3^0
\end{align*}
\]

\[
\begin{align*}
81 &= 1 \cdot 4^3 + 1 \cdot 4^2 + 1 \cdot 4^0 \\
    &= 1 \cdot 4^3 + 4 \cdot 4^1 + 1 \cdot 4^0 \\
    &= 4 \cdot 4^2 + 4 \cdot 4^1 + 1 \cdot 4^0
\end{align*}
\]
Adopting the notation for hyper $m$-ary partitions and the sets of these partitions, rewrite these partitions in the following way:

\[
H_3(37) = \{ [1, 1, 0, 1]_3, [1, 0, 3, 1]_3, [0, 3, 3, 1]_3 \};
\]

\[
H_4(81) = \{ [1, 1, 0, 1]_4, [1, 0, 4, 1]_4, [0, 4, 4, 1]_4 \}.
\]

Note that the number of hyper 3-ary partitions of 37 is the same as the number of hyper 4-ary partitions of 81. In other words,

\[
h_3(37) = h_4(F_{3,4}(37)) = h_4(81).
\]

We also observe that the coefficients of the partitions are similar, indicating that there is a relationship between the partitions in each set. This relationship will be further explored in the next section.

### 3 Bijects between hyper $m$-ary partitions and hyper $(m+1)$-ary partitions

We now verify the result suggested by the example in the prior section by considering hyper $m$-ary partitions of an integer $\ell$ and the hyper $(m+1)$-ary partitions of $k = F_{m,m+1}(\ell)$.

**Lemma 1.** For a positive integer $\ell$, let $k = F_{m,m+1}(\ell)$. Define $g_m : H_{m+1}(k) \rightarrow H_m(\ell)$ by mapping

\[
[c_r, c_{r-1}, \ldots, c_2, c_1, c_0]_{m+1} \mapsto [b_r, b_{r-1}, \ldots, b_2, b_1, b_0]_m
\]

according to the following rules:

\[
c_i = 0 \rightarrow b_i = 0
\]

\[
c_i = 1 \rightarrow b_i = 1
\]

\[\vdots\]

\[
c_i = m - 2 \rightarrow b_i = m - 2
\]

\[
c_i = m - 1 \rightarrow b_i = m - 1
\]

\[
c_i = m \rightarrow b_i = m - 1
\]

\[
c_i = m + 1 \rightarrow b_i = m.
\]

Then $g_m$ is a bijection.

**Proof.** It is clear from the definition that $g_m$ is a function. So we first show that $g_m$ is one-to-one. Suppose $x = [x_r, x_{r-1}, \ldots, x_2, x_1, x_0]_{m+1}$ and $y = [y_r, y_{r-1}, \ldots, y_2, y_1, y_0]_{m+1}$ are two hyper $(m+1)$-ary partitions of $k$ such that $x \neq y$. Then there must be at least one digit that doesn’t match. Let $J = \{j_1, j_2, \ldots, j_n\}$ be the set of indices such that $x_{j} \neq y_{j}$. Then we have two cases.
First suppose without loss of generality that there is an index \( j \) such that \( x_j \notin \{ m-1, m \} \). Then the \( j^{th} \) digit of \( g_m(x) \) will be different than the \( j^{th} \) digit of \( g_m(y) \). Thus \( g_m(x) \neq g_m(y) \).

Now suppose that \( x_j \in \{ m - 1, m \} \) and \( y_j \in \{ m - 1, m \} \) for all \( j \in J \). Let \( J_1 = \{ j \in J : x_j = m - 1 \} \) and \( J_2 = \{ j \in J : x_j = m \} \). Note that \( y_j = m \) for all \( j \in J_1 \) and \( y_j = m - 1 \) for all \( j \in J_2 \). Also observe that

\[
x = \sum_{j \notin J} x_j m^j + \sum_{j \in J_1} (m - 1)m^j + \sum_{j \in J_2} m \cdot m^j
\]

\[
y = \sum_{j \notin J} y_j m^j + \sum_{j \in J_1} m \cdot m^j + \sum_{j \in J_2} (m - 1)m^j.
\]

Since \( x_j = y_j \) for all \( j \notin J \),

\[
x - y = \sum_{j \in J_1} (m - 1 - m)m^j + \sum_{j \in J_2} (m - m + 1)m^j
\]

\[
= \sum_{j \in J_2} m^j - \sum_{j \in J_1} m^j.
\]

Observe that \( x - y = 0 \) since \( x \) and \( y \) are two different hyper \((m + 1)\)-ary partitions of the same number \( k \), implying

\[
\sum_{j \in J_2} m^j - \sum_{j \in J_1} m^j = 0.
\]

However, since \( J_1 \) and \( J_2 \) are disjoint, this is impossible. Thus it must be the case that when \( x \neq y \), one of \( x_j \) or \( y_j \) must be outside of \( \{ m - 1, m \} \) so that \( g_m(x) \neq g_m(y) \) as seen above. Thus \( g_m \) is one-to-one.

To show that \( g_m \) is onto, consider \( b = [b_r, b_{r-1}, \ldots, b_2, b_1, b_0]_m \in H_m(\ell) \). We then define \( c = [c_r, c_{r-1}, \ldots, c_2, c_1, c_0]_{m+1} \) in the following way. If \( b_i \in \{ 0, 1, 2, \ldots, m - 3, m - 2 \} \), then set \( c_i = b_i \) and if \( b_i = m \), set \( c_i = m + 1 \). Now suppose \( b_i = m - 1 \). Let \( v \) be the minimal index with \( v < i \) such that \( b_v \neq m - 1 \). If \( v \) does not exist, then set \( c_i = m - 1 \). If \( v \) does exist with \( b_v = m \), then set \( c_i = m \). If \( v \) exists with \( b_v \in \{ 0, 1, 2, \ldots, m - 2 \} \), then set \( c_i = m - 1 \). Notice that we may verify that \( c \in H_{m+1}(k) \) by converting \( c \) into the base \( m + 1 \) representation of \( k \). Therefore \( b \) is the image of \( c \) under \( g_m \) and thus \( g_m \) is onto.

This bijection implies that the number of \( m \)-ary partitions of any integer \( \ell \) is the same as the number of \((m + 1)\)-ary partitions of \( F_{m,m+1}(\ell) \).

## 4 Hyper \( m_1 \)-ary partitions and hyper \( m_2 \)-ary partitions

In this section, we use the result of Lemma 1 to define a more general bijection between \( H_{m_1}(n) \) and \( H_{m_2}(F_{m_1,m_2}(n)) \) for \( m_2 > m_1 + 1 \). To do this, we need the following lemma about hyper \( m_2 \)-ary partitions of an integer \( n \).
In the following proof, observe that multiplying a partition \([x_r, x_{r-1}, \ldots, x_2, x_1, x_0]_m\) by \(m\) corresponds to shifting the coefficients to the left one place and adding an additional 0 as the last coefficient.

**Lemma 2.** Let \(m_2 > m_1 + 1\). If the base \(m_2\) representation of an integer \(n\) contains only digits from the set \(\{0, 1, 2, \ldots, m_1 - 1\}\), then there are no hyper \(m_2\)-ary partitions of \(n\) which use any of the coefficients \(m_1, m_1 + 1, \ldots, m_2 - 2\).

**Proof.** We will prove this by induction on \(n\). Assume that for all \(q < n\), when the base \(m_2\) representation of \(q\) contains only digits from the set \(\{0, 1, 2, \ldots, m_1 - 1\}\), then there are no hyper \(m_2\)-ary partitions of \(n\) which use any of the coefficients \(m_1, m_1 + 1, \ldots, m_2 - 2\).

First, consider when \(n = m_2q\) and suppose that in base \(m_2\) the digits of \(n\) come from the set \(\{0, 1, 2, \ldots, m_1 - 1\}\). This means the digits in the base \(m_2\) representation of \(q\) also come only from this set. Now, apply the recurrence (1) to write \(h_{m_2}(m_2q) = h_{m_2}(q) + h_{m_2}(q - 1)\). This implies that every hyper \(m_2\)-ary partition of \(n\) is obtained from either a hyper \(m_2\)-ary partition of \(q\) or a hyper \(m_2\)-ary partition of \(q - 1\).

Observe that a hyper \(m_2\)-ary partition of \(n\) obtained from a hyper \(m_2\)-ary partition of \(q\) is found by multiplying the latter partition by \(m_2\), thereby shifting the coefficients of \(q\) and appending a 0 at the end. This results in hyper \(m_2\)-ary partitions of \(n\) whose coefficients are the same as the coefficients of hyper \(m_2\)-ary partitions of \(q\), along with an additional 0. Similarly, a hyper \(m_2\)-ary partition of \(n\) that is obtained from a hyper \(m_2\)-ary partition of \(q - 1\) is found by shifting the digits of the latter partition and appending an \(m_2\) to the end. This means we may write

\[
H_{m_2}(n) = \{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q)\} \\
\cup \{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0, m_2]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q - 1)\}.
\]

Since \(q - 1\) and \(q\) are less than \(n\), by the induction hypothesis we know the coefficients of all hyper \(m_2\)-ary partitions of \(q - 1\) and \(q\) are from the set \(\{0, 1, 2, \ldots, m_1 - 1, m_2 - 1, m_2\}\). Thus, the coefficients of any hyper \(m_2\)-ary partition of \(n\) are also from this set.

Now assume that \(n = m_2q + s\), where \(1 \leq s \leq m_2 - 1\). Observe that since the base \(m_2\) representation of \(n\) contains only digits from the set \(\{0, 1, 2, \ldots, m_1 - 1\}\), then we must have \(1 \leq s \leq m_1 - 1\). Furthermore, when \(n = m_2q + s\), apply the recurrence (2) to conclude that a hyper \(m_2\)-ary partition of \(n\) is obtained from a hyper \(m_2\)-ary partition of \(q\) by multiplying the latter partition by \(m_2\) and appending \(s\) to the end, where \(1 \leq s \leq m_1 - 1\). So,

\[
H_{m_2}(n) = \{[x_r, x_{r-1}, \cdots, x_2, x_1, x_0, s]_{m_2} : [x_r, x_{r-1}, \cdots, x_2, x_1, x_0]_{m_2} \in H_{m_2}(q)\}.
\]

Since \(q < n\), the coefficients of all hyper \(m_2\)-ary partitions of \(q\) are in the set \(\{0, 1, 2, \ldots, m_1 - 1, m_2 - 1, m_2\}\). Since \(s\) is an element of this set, we conclude that the coefficients of hyper \(m_2\)-ary partitions of \(n\) come from the same set.

Therefore, in all cases, the hyper \(m_2\)-ary partitions of \(n\) never contain any of the coefficients \(m_1, m_1 + 1, \ldots, m_2 - 2\).
Now we are ready to prove there is a bijection between hyper \( m_1 \)-ary partitions of an integer \( \ell \) and hyper \( m_2 \)-ary partitions of \( k = F_{m_1,m_2}(\ell) \).

**Lemma 3.** Let \( \ell \) be a positive integer and set \( k = F_{m_1,m_2}(\ell) \). Define \( \phi : H_{m_2}(k) \to H_{m_1}(\ell) \) by mapping

\[
[c_r, c_{r-1}, \ldots, c_2, c_1, c_0]_{m_2} \mapsto [b_r, b_{r-1}, \ldots, b_2, b_1, b_0]_{m_1}
\]

according to the following rules:

\[
\begin{align*}
c_i &= 0 \quad \rightarrow \quad b_i = 0 \\
\vdots \\
\end{align*}
\]

\[
\begin{align*}
c_i &= m_1 - 1 \quad \rightarrow \quad b_i = m_1 - 1 \\
c_i &= m_2 - 1 \quad \rightarrow \quad b_i = m_1 - 1 \\
c_i &= m_2 \quad \rightarrow \quad b_i = m_1.
\end{align*}
\]

Then, \( \phi \) is a bijection.

**Proof.** If \( m_2 = m_1 + 1 \), then the result follows immediately from Lemma 1. So, we assume that \( m_2 > m_1 + 1 \). From the definition of \( k \), we know the base \( m_2 \) representation of \( k \) includes only digits less than or equal to \( m_1 - 1 \). So, we apply Lemma 2 to conclude that none of the hyper \( m_2 \)-ary partitions in \( H_{m_2}(k) \) have any coefficients between \( m_1 \) and \( m_2 - 2 \), inclusive. Thus, \( \phi \) need only specify how to map coefficients from the set \( \{0, 1, \ldots, m_1 - 1, m_2 - 1, m_2\} \).

Now, using the bijection \( g_m \) given in Lemma 1, define a new function \( G : H_{m_2}(k) \to H_{m_1}(\ell) \) as follows:

\[ G = g_{m_1} \circ g_{m_1+1} \circ g_{m_1+2} \circ \cdots \circ g_{m_2-2} \circ g_{m_2-1}. \]

It is clear from Lemma 1 that when we apply \( G \) to any \( m_2 \)-ary partition coefficient which is less than or equal to \( m_1 - 1 \), the coefficient maps to itself. When we apply \( G \) to a partition coefficient of \( m_2 - 1 \), we see that

\[
m_2 - 1 \xrightarrow{g_{m_2-1}} m_2 - 2 \xrightarrow{g_{m_2-2}} m_2 - 3 \xrightarrow{g_{m_2-3}} \cdots \xrightarrow{g_{m_1}} m_1 - 1.
\]

Finally, when we apply \( G \) to a partition coefficient of \( m_2 \), we see that

\[
m_2 \xrightarrow{g_{m_2-1}} m_2 - 1 \xrightarrow{g_{m_2-2}} m_2 - 2 \xrightarrow{g_{m_2-3}} \cdots \xrightarrow{g_{m_1}} m_1.
\]

Thus, \( G = \phi \).

We have \( \phi \) equal to a finite composition of bijective functions. Therefore, \( \phi \) is a bijection. \( \square \)

Lemma 3 leads to the following identity between values of \( h_{m_1} \) and \( h_{m_2} \).
Theorem 4. Let \(2 \leq m_1 < m_2\). For positive integer \(\ell\), set \(k = F_{m_1, m_2}(\ell)\). Then
\[ h_{m_2}(k) = h_{m_1}(\ell). \]

Proof. The values \(\ell\) and \(k\) given here match Lemma 3 and we know that \(h_{m_1}(\ell) = |H_{m_1}(\ell)|\) and \(h_{m_2}(k) = |H_{m_2}(k)|\). Lemma 3 gives a bijection between these finite sets. Therefore, we conclude that the sets must have the same cardinality. \(\square\)

As an immediate corollary, we now state a final result regarding the relationships between hyper \(m\)-ary partition sequences for different values of \(m\).

Corollary 5. Let \(2 \leq m_1 \leq m_2\). Then \(h_{m_1}\) is a subsequence of \(h_{m_2}\).

These theorems extend the results in [4], ultimately showing that the subsequence identity holds for any hyper \(m_1\)-ary and hyper \(m_2\)-ary partition sequences.

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References


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