Arithmetic Properties of Partition $k$-tuples with Odd Parts Distinct

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Abstract
Let $\text{pod}_k(n)$ denote the number of partition $k$-tuples of $n$ wherein odd parts are distinct (and even parts are unrestricted). We establish some interesting infinite families of congruences and internal congruences modulo 4, 16, and 5 for $\text{pod}_2(n)$, $\text{pod}_4(n)$, and $\text{pod}_6(n)$, respectively. We also find Ramanujan-type congruences modulo 5 for $\text{pod}_3(n)$ and densities of $\text{pod}_2(n)$, $\text{pod}_3(n)$, $\text{pod}_4(n)$, and $\text{pod}_6(n)$ modulo 4, 5, 16, and 5, respectively.

1 Introduction

For $|q| < 1$, Ramanujan’s theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty} (q^2; q^2)_{\infty}$$

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\[
\psi(q) := \sum_{n=0}^{\infty} q^{(n^2+n)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2+n} = \frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}},
\]
(2)

where \((a;q)_{\infty} = (1-a)(1-aq)(1-aq^2)\cdots\).

Let \(\text{pod}(n)\) denote the number of partitions of \(n\) wherein odd parts are distinct (and even parts are unrestricted). The generating function of \(\text{pod}(n)\) is

\[
\sum_{n=0}^{\infty} \text{pod}(n)q^n = \frac{(-q;q^2)_{\infty}}{(q^2;q^2)_{\infty}} = \frac{1}{\psi(-q)}.
\]

In 2010, Hirschhorn and Sellers [5] proved that, for all \(\alpha \geq 0\) and \(n \geq 0\),

\[
\text{pod}\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod{3}.
\]

They also found some internal congruences such as

\[
\text{pod}(81n + 17) \equiv 5\text{pod}(9n + 2) \pmod{27}.
\]

Recently, Wang [10] established new congruences for \(\text{pod}(n)\). For example, for each \(\alpha \geq 1\) and \(n \geq 0\),

\[
\text{pod}\left(5^{2\alpha+2}n + \frac{11 \times 5^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod{5}.
\]

Let \(\text{pod}_{-k}(n)\) denote the number of partition \(k\)-tuples of \(n\) wherein odd parts are distinct (and even parts are unrestricted). The generating function of \(\text{pod}_{-k}(n)\) is

\[
\sum_{n=0}^{\infty} \text{pod}_{-k}(n)q^n = \frac{(-q;q^2)^k_{\infty}}{(q^2;q^2)^k_{\infty}} = \frac{1}{\psi(-q)^k}.
\]

(3)

Chen and Lin [3] established congruences modulo 3 and 5 for \(\text{pod}_{-2}(n)\). For example, for \(\alpha \geq 1\) and \(n \geq 0\),

\[
\text{pod}_{-2}\left(5^{\alpha+1}n + \frac{11 \times 5^{\alpha} + 1}{4}\right) \equiv 0 \pmod{5}.
\]

Wang [8, 9] has established congruences modulo 7, 9, and 11 satisfied by \(\text{pod}_{-3}(n)\) and congruences modulo 5, 9, and 81 satisfied by \(\text{pod}_{-4}(n)\) by employing theta function identities. For example, for \(\alpha \geq 1\) and \(n \geq 0\),

\[
\text{pod}_{-3}\left(3^{2\alpha+2}n + \frac{23 \times 3^{2\alpha+1} + 3}{8}\right) \equiv 0 \pmod{9}
\]

and

\[
\text{pod}_{-4}\left(3^{\alpha+1}n + \frac{5 \times 3^{\alpha} + 1}{2}\right) \equiv 0 \pmod{9}.
\]
He also found some internal congruences such as
\[ \text{pod}_4(27n + 5) \equiv -\text{pod}_4(9n + 2) \pmod{9}. \]

In this paper, we establish congruences modulo powers of 2 and modulo 5 for \(\text{pod}_k(n)\) for \(k \in \{2, 3, 4, 6\}\). In this vein, in Section 3, we find infinite family of congruences and internal congruences modulo 4 satisfied by \(\text{pod}_2(n)\) and we also find density of \(\text{pod}_2(n)\) modulo 4. In Section 4, we prove Ramanujan-type congruences modulo 5 for \(\text{pod}_3(n)\) and that \(\text{pod}_3(n)\) is divisible by 5 at least \(1/30\) of the time. In Section 5, we establish infinite family of congruences and internal congruences modulo 16 satisfied by \(\text{pod}_4(n)\) following density of \(\text{pod}_4(n)\) modulo 16. In Section 6, we determine infinite family of congruences and internal congruences modulo 5 satisfied by \(\text{pod}_6(n)\) and we also determine density of \(\text{pod}_6(n)\) modulo 5.

2 Preliminaries

The following results are useful in proving our main results.

**Lemma 1.** [2, pp. 40–49] We have

\[
\varphi(q) = \varphi(q^4) + 2q\psi(q^8),
\]

\[
\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2,
\]

\[
\psi(q) = f(q^3, q^6) + q\psi(q^9)
\]

\[
= f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}),
\]

\[
\psi(q)^2 = \varphi(q)\psi(q^2).
\]

**Lemma 2.** [1, Eq. 1.6.7, p. 26] We have

\[
f(q, q^4)f(q^2, q^3) = \psi(q)^2 - q\psi(q^5)^2.
\]

**Lemma 3.** Let \(\sum_{n=0}^{\infty} h(n)q^n = q\psi(q)^4\). Then

\[
\sum_{n=0}^{\infty} h(5n + 3)q^n \equiv \psi(q)^4 \pmod{5}.
\]
Proof. From (7), it follows that
\[
\sum_{n=0}^{\infty} h(n)q^n = q\psi(q)^4 \\
= q \left( f(q^{10}, q^{15}) + q f(q^5, q^{20}) + q^3 \psi(q^{25}) \right)^4 \\
= 12q^5 f(q^{10}, q^{15})^2 f(q^5, q^{20}) \psi(q^{25}) + 4q^{10} f(q^{10}, q^{15})\psi(q^{25})^3 \\
+ 12q^8 f(q^{10}, q^{15}) f(q^5, q^{20}) \psi(q^{25})^2 + 6q^7 f(q^{10}, q^{15})^2 \psi(q^{25})^2 \\
+ 4q^2 f(q^{10}, q^{15})^3 f(q^5, q^{20}) + 4q^4 f(q^{10}, q^{15})^3 \psi(q^{25}) \\
+ 6q^3 f(q^{10}, q^{15})^2 f(q^5, q^{20})^2 + 5q^5 f(q^5, q^{20})^4 + q^{13} \psi(q^{25})^4 \\
+ 4q^4 f(q^{10}, q^{15}) f(q^5, q^{20})^3 + 6q^3 f(q^5, q^{20})^2 \psi(q^{25})^2 \\
+ 4q^7 f(q^5, q^{20})^3 \psi(q^{25}) + 12q^6 f(q^{10}, q^{15}) f(q^5, q^{20})^2 \psi(q^{25}) \\
+ 4q^{11} f(q^5, q^{20}) \psi(q^{25})^3 + q f(q^{10}, q^{15})^4,
\]
which yields
\[
\sum_{n=0}^{\infty} h(5n+3)q^n = 2q f(q^2, q^3) f(q, q^4) \psi(q^5)^2 + f(q^2, q^3)^2 f(q, q^4)^2 \\
+ q^2 \psi(q^5)^4 \pmod{5}.
\]
Using (9) in the above equation, we arrive at (10).

Lemma 4. Let \( \sum_{n=0}^{\infty} g(n)q^n = \psi(q)^2. \) Then

\[
\sum_{n=0}^{\infty} g(5n+1)q^n = 2\psi(q)^2 - q\psi(q^5)^2.
\] (11)

Proof. It follows from (7) that
\[
\sum_{n=0}^{\infty} g(n)q^n = \psi(q)^2 \\
= \left( f(q^{10}, q^{15}) + q f(q^5, q^{20}) + q^3 \psi(q^{25}) \right)^2 \\
= q^6 \psi(q^{25})^2 + 2q^4 \psi(q^{25}) f(q^5, q^{20}) + 2q^3 \psi(q^{25}) f(q^{10}, q^{15}) \\
+ q^2 f(q^5, q^{20})^2 + 2q f(q^{10}, q^{15}) f(q^5, q^{20}) + f(q^{10}, q^{15})^2,
\]
which yields
\[
\sum_{n=0}^{\infty} g(5n+1)q^n = 2f(q^2, q^3) f(q, q^4) + q\psi(q^5)^2.
\]
Using (9) in the above equation, we arrive at (11).
Lemma 5. [4, Theorem 2.1] For any odd prime, $p$,

$$\psi(q) = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f \left( q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}} \right) + q^{\frac{p^2-1}{8}} \psi(q^2). \quad (12)$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$, for $0 \leq m \leq \frac{p-3}{2}$.

3 Arithmetic properties of $\pod_{-2}(n)$

In this section, we prove the infinite family of congruences and internal congruences modulo 4 for $\pod_{-2}(n)$.

3.1 Infinite family of congruences modulo 4

Theorem 6. Let $p$ be any odd prime such that $\left(\frac{-2}{p}\right) = -1$ and $\alpha \geq 0$. Then

$$\sum_{n=0}^{\infty} \pod_{-2} \left( 2p^{2\alpha}n + \frac{3p^{2\alpha}+1}{4} \right) q^n \equiv 2\psi(q)\psi(q^2) \pmod{4} \quad (13)$$

and, for all $n \geq 0$ and $1 \leq \xi \leq p - 1$,

$$\pod_{-2} \left( 2p^{2\alpha+1}(pn + \xi) + \frac{3p^{2\alpha+2}+1}{4} \right) \equiv 0 \pmod{4}. \quad (14)$$

Proof. We have

$$\sum_{n=0}^{\infty} \pod_{-2}(n)q^n = \frac{1}{\psi(-q)^2}. \quad (15)$$

Invoking (8) and (15),

$$\sum_{n=0}^{\infty} \pod_{-2}(n)q^n = \frac{1}{\psi(q^2)\varphi(-q)}$$

$$= \frac{(1 - (1 - \varphi(-q)))^{-1}}{\psi(q^2)}$$

$$= \frac{1 + (1 - \varphi(-q)) + (1 - \varphi(-q))^2 + \cdots}{\psi(q^2)}$$

$$\equiv \frac{2 - \varphi(-q)}{\psi(q^2)} \pmod{4} \quad \text{from (1).}$$

5
Using (4) in the above equation, we find that
\[
\sum_{n=0}^{\infty} \text{pod}_2(n)q^n \equiv 2 - \varphi(q^4) + 2q\psi(q^8) \pmod{4},
\]
which yields
\[
\sum_{n=0}^{\infty} \text{pod}_2(2n + 1)q^n \equiv 2\frac{\psi(q^4)}{\psi(q)} \pmod{4}. \tag{16}
\]
From the binomial theorem, we can see that for any prime \(p\) and for each positive integer \(\ell\),
\[
(q; q)^{p\ell} \equiv (q^p; q^p)^{p\ell-1} \pmod{p^\ell}. \tag{17}
\]
In view of (17), (16) can be expressed as
\[
\sum_{n=0}^{\infty} \text{pod}_2(2n + 1)q^n \equiv 2\psi(q)\psi(q^2) \pmod{4}, \tag{18}
\]
which is the \(\alpha = 0\) case of (13). If we assume that (13) holds for some \(\alpha \geq 0\), then, substituting (12) in (13),
\[
\sum_{n=0}^{\infty} \text{pod}_2 \left( 2p^{2\alpha}n + \frac{3p^{2\alpha} + 1}{4} \right)q^n
\]
\[
\equiv 2 \left( \sum_{m=0}^{p-3} q^{m^2+m}f \left( q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}} \right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \right)
\]
\[
\times \left( \sum_{m=0}^{p-3} q^{m^2+m}f \left( q^{p^2+(2m+1)p}, q^{p^2-(2m+1)p} \right) + q^{\frac{p^2-1}{8}}\psi(q^{2p^2}) \right) \pmod{4}. \tag{19}
\]
For any odd prime, \(p\), and \(0 \leq m_1, m_2 \leq (p - 3)/2\), consider the congruence
\[
\frac{m_1^2 + m_1}{2} + 2 \times \frac{m_2^2 + m_2}{2} \equiv \frac{3p^2 - 3}{8} \pmod{p},
\]
which implies that
\[
(2m_1 + 1)^2 + 2(2m_2 + 1)^2 \equiv 0 \pmod{p}. \tag{20}
\]
Since \(\left( \frac{-2}{p} \right) = -1\), the only solution of the congruence (20) is \(m_1 = m_2 = \frac{p - 1}{2}\). Therefore, equating the coefficients of \(q^{p\ell + \frac{3p^2 - 3}{8}}\) from both sides of (19), dividing throughout by \(q^{\frac{3p^2 - 3}{8}}\) and then replacing \(q^p\) by \(q\), we obtain
\[
\sum_{n=0}^{\infty} \text{pod}_2 \left( 2p^{2\alpha+1}n + \frac{3p^{2\alpha+2} + 1}{4} \right)q^n \equiv 2\psi(q^p)\psi(q^{2p}) \pmod{4}. \tag{21}
\]
Equating the coefficients of $q^{pn}$ on both sides of (21) and then replacing $q^p$ by $q$, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_2 \left( 2p^{2\alpha+2}n + \frac{3p^{2\alpha+2} + 1}{4} \right) q^n \equiv 2\psi(q)\psi(q^2) \pmod{4},$$

which is the $\alpha + 1$ case of (13).

Equating the coefficients of $q^{pn+\xi}$ for $1 \leq \xi \leq p - 1$ from (21), we arrive at (14).

\[ \square \]

**Corollary 7.** Let $p$ be any odd prime such that $\left( \frac{-2}{p} \right) = -1$. Then $\mathrm{pod}_2(n)$ is divisible by 4 for at least $\frac{1}{2(p+1)}$ of all nonnegative integers $n$.

**Proof.** The arithmetic sequences $\left\{ 2p^{2\alpha+1}(pn + \xi) + \frac{3p^{2\alpha+2} + 1}{4} : \alpha \geq 0 \right\}$ for $1 \leq \xi \leq p - 1$, on which $\mathrm{pod}_2(\cdot)$ is 0 modulo 4, do not intersect. These sequences account for

$$(p - 1) \left( \frac{1}{2p^2} + \frac{1}{2p^4} + \frac{1}{2p^6} + \cdots \right) = \frac{1}{2(p+1)}$$

of all nonnegative integers. \[ \square \]

### 3.2 Some internal congruences

**Theorem 8.** For each $n \geq 0$,

\[
\begin{align*}
\mathrm{pod}_2(54n + 25) & \equiv \mathrm{pod}_2(6n + 3) \pmod{4}, \\
\mathrm{pod}_2(54n + 43) & \equiv \mathrm{pod}_2(6n + 5) \pmod{4}, \\
\mathrm{pod}_2(162n + 7) & \equiv 2\mathrm{pod}_2(18n + 1) \pmod{4}, \\
\mathrm{pod}_2(162n + 115) & \equiv 2\mathrm{pod}_2(18n + 13) \pmod{4}.
\end{align*}
\]

**Proof.** If $\sum_{n=0}^{\infty} a(n)q^n = \psi(q)\psi(q^2)$, then the authors [6] found that

$$\sum_{n=0}^{\infty} a(3n)q^n = \psi(q)\varphi(q) - q\psi(q^3)\psi(q^6).$$

Using (26), we can express (18) as

$$\sum_{n=0}^{\infty} \mathrm{pod}_2(6n + 1)q^n \equiv 2\psi(q)\varphi(q) - 2q\psi(q^3)\psi(q^6) \pmod{4}. \quad (27)$$

Invoking (1) and (27),

$$\sum_{n=0}^{\infty} \mathrm{pod}_2(6n + 1)q^n \equiv 2\psi(q) + 2q\psi(q^3)\psi(q^6) \pmod{4}. \quad (28)$$
Substituting (6) into (28) and extracting the terms involving $q^{3n+1}$,

$$\sum_{n=0}^{\infty} \text{pod}_{-2}(18n + 7)q^n \equiv 2\psi(q^3) + 2\psi(q)\psi(q^2) \pmod{4},$$

which is equivalent to

$$\sum_{n=0}^{\infty} \text{pod}_{-2}(18n + 7)q^n \equiv 2\psi(q^3) + \sum_{n=0}^{\infty} \text{pod}_{-2}(2n + 1)q^n \pmod{4}. \quad (29)$$

Equating the coefficients of $q^{3n+1}$ and $q^{3n+2}$ from (29), we arrive at (22) and (23), respectively. Equating the coefficients of $q^{3n}$ and then replacing $q^3$ by $q$ from (29),

$$\sum_{n=0}^{\infty} \text{pod}_{-2}(54n + 7)q^n \equiv 2\psi(q) + \sum_{n=0}^{\infty} \text{pod}_{-2}(6n + 1)q^n \pmod{4}. \quad (30)$$

Invoking (28) and (30),

$$\sum_{n=0}^{\infty} \text{pod}_{-2}(54n + 7)q^n \equiv 2\sum_{n=0}^{\infty} \text{pod}_{-2}(6n + 1)q^n - 2q\psi(q^3)\psi(q^5) \pmod{4}, \quad (31)$$

Equating the coefficients of $q^{3n}$ and $q^{3n+2}$ from (31), we arrive at (24) and (25), respectively. \hfill \Box

## 4 Ramanujan-type congruences for pod$_{-3}(n)$

In this section, we prove the Ramanujan-type congruences modulo 5 for pod$_{-3}(n)$.

**Theorem 9.** For each $\alpha \geq 1$,

$$\text{pod}_{-3}\left(5^{2\alpha+1}n + \frac{\mu \times 5^{2\alpha} + 3}{8}\right) \equiv 0 \pmod{5}, \quad (32)$$

where $\mu = 13, 21, 29,$ and $37$.

**Proof.** We have

$$\sum_{n=0}^{\infty} \text{pod}_{-3}(n)(-1)^n q^n = \frac{1}{\psi(q)^3}.$$

It follows from (17) that

$$\sum_{n=0}^{\infty} \text{pod}_{-3}(n)(-1)^n q^n \equiv \frac{\psi(q)^2}{\psi(q^5)} \pmod{5}
\equiv \frac{1}{\psi(q^5)} \sum_{n=0}^{\infty} g(n)q^n \pmod{5}.$$
Extracting the terms involving $q^{5n+1}$, dividing throughout by $q$ and then replacing $q^5$ by $q$,
\[
\sum_{n=0}^{\infty} \text{pod}_-3(5n+1)(-1)^{n+1}q^n \equiv \frac{1}{\psi(q)} \sum_{n=0}^{\infty} g(5n+1)q^n \pmod{5}
\equiv \frac{1}{\psi(q)} (2\psi(q)^2 - q\psi(q^5)^2) \pmod{5} \text{ from (11)}
\equiv 2\psi(q) - q\psi(q^5)\psi(q) \pmod{5} \text{ using (17).} \quad (33)
\]
Substituting (7) into (33) and from the Lemma (3), we find that
\[
\sum_{n=0}^{\infty} \text{pod}_-3(5n+1)(-1)^{n+1}q^n \equiv 2f(q^{10}, q^{15}) + 2qf(q^5, q^{20}) + 2q^3\psi(q^{25}) - \psi(q^5) \sum_{n=0}^{\infty} h(n)q^n \pmod{5},
\]
which implies that
\[
\sum_{n=0}^{\infty} \text{pod}_-3(25n+16)(-1)^nq^n \equiv 2\psi(q^5) - \psi(q) \sum_{n=0}^{\infty} h(5n+3)q^n \pmod{5}
\equiv 2\psi(q^5) - \psi(q)^5 \pmod{5} \text{ using (10).} \quad (34)
\]
Using (17), (34) can be expressed as
\[
\sum_{n=0}^{\infty} \text{pod}_-3(25n+16)(-1)^nq^n \equiv \psi(q^5) \pmod{5}. \quad (35)
\]
Extracting the terms involving $q^{5n}$ from (35),
\[
\sum_{n=0}^{\infty} \text{pod}_-3(125n+16)(-1)^nq^n \equiv \psi(q) \pmod{5}
\equiv f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}) \pmod{5},
\]
which yields
\[
\sum_{n=0}^{\infty} \text{pod}_-3(625n+391)(-1)^nq^n \equiv \psi(q^5) \pmod{5}. \quad (36)
\]
From (35), (36), and by induction, we find that for each $\alpha \geq 1$,
\[
\sum_{n=0}^{\infty} \text{pod}_-3 \left(5^{2\alpha}n + \frac{5^{2\alpha+1}+3}{8}\right)(-1)^{n+1}q^n \equiv (-1)^\alpha\psi(q^5) \pmod{5}. \quad (37)
\]
Equating the coefficients of $q^{5n+\xi}$ for $1 \leq \xi \leq 4$ from (37), we arrive at (32). \qed
Corollary 10. The function $\text{pod}_3(n)$ is divisible by 5 for at least $\frac{1}{30}$ of all nonnegative integers $n$.

Proof. The arithmetic sequences $\left\{ 5^{2\alpha+1}n + \frac{\mu \times 5^{2\alpha+3}}{8} : \alpha \geq 1 \right\}$ for $\mu = 13, 21, 29, \text{and} \ 37$, on which $\text{pod}_3(\cdot)$ is 0 modulo 5, do not intersect. These sequences account for
\[
4 \left( \frac{1}{5^3} + \frac{1}{5^5} + \frac{1}{5^7} + \cdots \right) = \frac{1}{30}
\]
of all nonnegative integers. \qed

5 Arithmetic properties of $\text{pod}_4(n)$

In this section, we prove the infinite family of congruences and internal congruences modulo 16 for $\text{pod}_4(n)$.

5.1 Infinite family of congruences modulo 16

Theorem 11. Let $p$ be any prime such that $p \equiv 3 \pmod{4}$ and $\alpha \geq 0$. Then
\[
\sum_{n=0}^{\infty} \text{pod}_4 \left( 2p^{2\alpha}n + \frac{p^{2\alpha}+1}{2} \right) q^n \equiv 4\psi(q)^2 \pmod{16} \quad (38)
\]
and, for all nonnegative integers $n$ and $1 \leq \xi \leq p - 1$,
\[
\text{pod}_4 \left( 2p^{2\alpha+1}(pn + \xi) + \frac{p^{2\alpha+2}+1}{2} \right) \equiv 0 \pmod{16}. \quad (39)
\]

Proof. We have
\[
\sum_{n=0}^{\infty} \text{pod}_4(n)q^n = \frac{1}{\psi(-q)^4}. \quad (40)
\]
Invoking (8) and (40),
\[
\sum_{n=0}^{\infty} \text{pod}_4(n)q^n = \frac{1}{\psi(q^2)^2 \varphi(-q)^2} = \frac{(1 - \varphi(-q)^2))^{-1}}{\psi(q^2)^2} = \frac{1 + (1 - \varphi(-q)^2) + (1 - \varphi(-q)^2)^2 + \cdots}{\psi(q^2)^2} \equiv \frac{2 - \varphi(-q)^2}{\psi(q^2)^2} \pmod{16} \text{ using (1)}
\]
\[
\equiv \frac{2 - \varphi(q^2)^2 + 4q\psi(q^2)^2}{\psi(q^2)^2} \pmod{16} \text{ from (5)},
\]
which implies that
\[
\sum_{n=0}^{\infty} \text{pod}_{-4}(2n+1)q^n \equiv 4\frac{\psi(q^2)^2}{\psi(q)^2} \pmod{16},
\] (41)

In view of (17), (41) can be expressed as
\[
\sum_{n=0}^{\infty} \text{pod}_{-4}(2n+1)q^n \equiv 4\psi(q)^2 \pmod{16},
\] (42)

which is the \( \alpha = 0 \) case of (38). If we assume that (38) holds for some \( \alpha \geq 0 \), then, substituting (12) into (38),
\[
\sum_{n=0}^{\infty} \text{pod}_{-4} \left(2p^{2\alpha}n + \frac{p^{2\alpha} + 1}{2}\right) q^n \\
\equiv 4 \left( \sum_{m=0}^{\frac{p-1}{2}} q^{\frac{p^2 - (2m + 1)p}{2}} + q^{\frac{p^2}{8} - \psi(q^2)} \right)^2.
\] (43)

For any odd prime, \( p \), and \( 0 \leq m_1, m_2 \leq (p - 3)/2 \), consider the congruence
\[
\frac{m_1^2 + m_1}{2} + \frac{m_2^2 + m_2}{2} \equiv \frac{2p^2 - 2}{8} \pmod{p},
\]
which implies that
\[
(2m_1 + 1)^2 + (2m_2 + 1)^2 \equiv 0 \pmod{p}. \tag{44}
\]

Since \( \left( \frac{-1}{p} \right) = -1 \) for \( p \equiv 3 \pmod{4} \), the only solution of the congruence (44) is \( m_1 = m_2 = \frac{p - 1}{2} \). Therefore, equating the coefficients of \( q^{m_1 + 2p^{2\alpha + 2}/8} \) from both sides of (43), dividing throughout by \( q^{2p^{2\alpha + 2}/8} \) and then replacing \( q^p \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} \text{pod}_{-4} \left(2p^{2\alpha} \left(pn + \frac{2p^2 - 2}{8}\right) + \frac{p^{2\alpha} + 1}{2}\right) q^n \equiv 4\psi(q^p)^2 \pmod{16}.
\] (45)

Equating the coefficients of \( q^{m_1} \) on both sides of (45) and then replacing \( q^p \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} \text{pod}_{-4} \left(2p^{2\alpha + 2}n + \frac{p^{2\alpha + 2} + 1}{2}\right) q^n \equiv 4\psi(q)^2 \pmod{16},
\]
which is the \( \alpha + 1 \) case of (38).

Equating the coefficients of \( q^{m_1 + \xi} \) for \( 1 \leq \xi \leq p - 1 \) from (45), we arrive at (39). □
Corollary 12. Let $p$ be a prime such that $p \equiv 3 \pmod{4}$. Then $\text{pod}_4(n)$ is divisible by 16 for at least $\frac{1}{2(p+1)}$ of all nonnegative integers $n$.

Proof. The arithmetic sequences $\left\{2p^{2\alpha+1}(pn + \xi) + \frac{p^{2\alpha+2}+1}{2} : \alpha \geq 0 \right\}$ for $1 \leq \xi \leq p - 1$, on which $\text{pod}_4(\cdot)$ is 0 modulo 16, do not intersect. These sequences account for

$$(p-1) \left(\frac{1}{2p^2} + \frac{1}{2p^4} + \frac{1}{2p^6} + \cdots \right) = \frac{1}{2(p+1)}$$

of all nonnegative integers.

5.2 Some internal congruences

Theorem 13. For each $n \geq 0$,

$$\text{pod}_4(50n + 3) \equiv 2\text{pod}_4(10n + 1) \pmod{16},$$
$$\text{pod}_4(50n + 23) \equiv 2\text{pod}_4(10n + 5) \pmod{16},$$
$$\text{pod}_4(50n + 33) \equiv 2\text{pod}_4(10n + 7) \pmod{16},$$
$$\text{pod}_4(50n + 43) \equiv 2\text{pod}_4(10n + 9) \pmod{16}.$$

Proof. If $\sum_{n=0}^{\infty} g(n)q^n = \psi(q)^2$, then (42) can be expressed as

$$\sum_{n=0}^{\infty} \text{pod}_4(2n + 1)q^n \equiv 4 \sum_{n=0}^{\infty} g(n)q^n \pmod{16},$$
which yields

$$\sum_{n=0}^{\infty} \text{pod}_4(10n + 3)q^n \equiv 4 \sum_{n=0}^{\infty} g(5n + 1)q^n \pmod{16}. \quad (46)$$

Invoking (11) and (46),

$$\sum_{n=0}^{\infty} \text{pod}_4(10n + 3)q^n \equiv 8\psi(q)^2 - 4q\psi(q^5)^2 \pmod{16}. \quad (47)$$

Substituting (42) into (47),

$$\sum_{n=0}^{\infty} \text{pod}_4(10n + 3)q^n \equiv 2 \sum_{n=0}^{\infty} \text{pod}_4(2n + 1)q^n - 4q\psi(q^5)^2 \pmod{16},$$
equating the coefficients of $q^{5n+i}$ for $i = 0, 2, 3, \text{ and } 4$ from the above equation, we obtain the desired results. \qed
6 Arithmetic properties of pod\(_{-6}(n)\)

In this section, we prove the infinite family of congruences and internal congruences modulo 5 for pod\(_{-6}(n)\).

6.1 Infinite family of congruences modulo 5

Theorem 14. Let \(p\) be any prime such that \(p \equiv 3 \pmod{4}\) and \(\alpha \geq 0\). Then

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(5p^{2\alpha}n + \frac{5p^{2\alpha} + 3}{4})q^n \equiv \psi(q)^2 \pmod{5}
\]

and, for each \(n \geq 0\) and \(1 \leq \xi \leq p - 1\),

\[
\text{pod}_{-6}\left(5p^{2\alpha+1}(pn + \xi) + \frac{5p^{2\alpha+2} + 3}{4}\right) \equiv 0 \pmod{5}.
\]

Proof. We have

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(n)q^n = \frac{1}{\psi(-q)^6}. \tag{48}
\]

In view of (17), (48) can be expressed as

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(n)(-1)^nq^n \equiv \frac{\psi(q)^4}{\psi(q^5)^2} \pmod{5}. \tag{49}
\]

Substituting (7) into (49),

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(n)(-1)^nq^n \equiv \frac{(f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}))^4}{\psi(q^5)^2} \pmod{5},
\]

which yields

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(5n+2)(-1)^nq^n \equiv \frac{f(q^2, q^3)^2f(q, q^4)^2}{\psi(q)^2} + \frac{2qf(q^2, q^3)f(q, q^4)\psi(q^5)^2}{\psi(q)^2} + \frac{q^2\psi(q^5)^4}{\psi(q)^2} \pmod{5}. \tag{50}
\]

Invoking (9) and (50),

\[
\sum_{n=0}^{\infty} \text{pod}_{-6}(5n+2)(-1)^nq^n \equiv \psi(q)^2 + q^2\psi(q^5)^4\psi(q)^2 - 2q\psi(q^5)^2 + 2q\psi(q^5)^2 - 2q^2\psi(q^5)^4\psi(q)^2 + q^2\psi(q^5)^4 \pmod{5}, \tag{51}
\]

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which implies that
\[
\sum_{n=0}^{\infty} \text{pod}_6(5n+2)(-1)^{n}q^n \equiv \psi(q)^2 \pmod{5}. \tag{52}
\]
The remainder of the proof is similar to that of Theorem 11, but rather than (42), we use (52).

Corollary 15. Let \( p \) be a prime such that \( p \equiv 3 \pmod{4} \). Then \( \text{pod}_6(n) \) is divisible by 5 for at least \( \frac{1}{5(p+1)} \) of all nonnegative integers \( n \).

Proof. The arithmetic sequences \( \left\{ 5p^{2\alpha+1}(pn+\xi)+\frac{5p^{2\alpha+2}+3}{4} : \alpha \geq 0 \right\} \) for \( 1 \leq \xi \leq p-1 \), on which \( \text{pod}_6(\cdot) \) is 0 modulo 5, do not intersect. These sequences account for
\[
(p-1) \left( \frac{1}{5p^2} + \frac{1}{5p^4} + \frac{1}{5p^6} + \cdots \right) = \frac{1}{5(p+1)}
\]
of all nonnegative integers. \( \square \)

6.2 Some internal congruences

Theorem 16. For each \( n \geq 0 \),
\[
\begin{align*}
\text{pod}_6(125n+7) & \equiv 3\text{pod}_6(25n+2) \pmod{5}, \\
\text{pod}_6(125n+57) & \equiv 3\text{pod}_6(25n+12) \pmod{5}, \\
\text{pod}_6(125n+82) & \equiv 3\text{pod}_6(25n+17) \pmod{5}, \\
\text{pod}_6(125n+107) & \equiv 3\text{pod}_6(25n+22) \pmod{5}.
\end{align*}
\]

Proof. The proof is similar to that of Theorem 13, but rather than (42), we use (52). \( \square \)

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References


[10] L. Wang, New congruences for partitions where the odd parts are distinct, *J. Integer Sequences* (2015), Article 15.4.2.

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