Tribonacci Numbers and the Brocard-Ramanujan Equation

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Abstract

Let \((T_n)_{n \geq 0}\) be the Tribonacci sequence defined by the recurrence \(T_{n+2} = T_{n+1} + T_n + T_{n-1}\), with \(T_0 = 0\) and \(T_1 = T_2 = 1\). In this short note, we prove that there are no integer solutions \((u, m)\) to the Brocard-Ramanujan equation \(m! + 1 = u^2\) where \(u\) is a Tribonacci number.

1 Introduction

In the past few years, several authors have considered Diophantine equations involving factorial numbers. For instance, Erdős and Selfridge [6] proved that \(n!\) is a perfect power only when \(n = 1\). However, the most famous among such equations was posed by Brocard [5] in 1876 and independently by Ramanujan ([17], [18, p. 327]) in 1913. The Diophantine equation

\[ m! + 1 = u^2 \]  

is now known as Brocard-Ramanujan Diophantine equation.

It is a simple matter to find the three known solutions, namely \(m = 4, 5\) and \(7\). Recently, Berndt and Galway [2] showed that there are no further solutions with \(m \leq 10^9\). An interesting contribution to the problem is due to Overholt [15], who showed that the equation
has only finitely many solutions if we assume a weak version of the abc conjecture. However, the Brocard-Ramanujan equation is still an open problem.

Let \((F_n)_{n \geq 0}\) be the Fibonacci sequence (sequence \texttt{A000045} in the OEIS \([19]\)) given by \(F_0 = 0, F_1 = 1\) and \(F_{n+2} = F_{n+1} + F_n\), for \(n \geq 0\).

A number of mathematicians have been interested in Diophantine equations that involve both factorial and Fibonacci numbers. For example, Grossman and Luca \([8]\) showed that if \(k\) is fixed, then there are only finitely many positive integers \(n\) such that

\[ F_n = m_1! + m_2! + \cdots + m_k! \]

holds for some positive integers \(m_1, \ldots, m_k\). Also, all the solutions for the case \(k \leq 2\) were determined. Later, Bollman, Hernández and Luca \([3]\) solved the case \(k = 3\). In a recent paper, Luca and Siksek \([11]\) found all factorials expressible as the sum of at least three Fibonacci numbers.

In 1999, Luca \([10]\) proved that \(F_n\) is a product of factorials only when \(n = 1, 2, 3, 6\) and 12. Also, Luca and Stănică \([12]\) showed that the largest product of distinct Fibonacci numbers which is a product of factorials is \(F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!\).

In 2012, Marques \([13]\) proved that \((m, u) = (4, 5)\) is the only solution of Eq. (1) where \(u\) is a Fibonacci number. His proof depends on the primitive divisor theorem together with factorizations formulas for \(F_n \pm 1\).

Among the several generalizations of Fibonacci numbers, one of the best known is the Tribonacci sequence \((T_n)_{n \geq 0}\) (sequence \texttt{A000073} in the OEIS). This is defined by the recurrence \(T_{n+1} = T_n + T_{n-1} + T_{n-2}\), with initial values \(T_0 = 0\) and \(T_1 = T_2 = 1\). The first few terms of this sequence are

\[ 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705. \]

Tribonacci numbers have a long history. They were first studied in 1914 by Agronomof \([1]\) and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg \([7]\).

Here, we are interested in solutions \((m, u)\) of the Brocard-Ramanujan equation where \(u\) is a Tribonacci number. We point out that in this we have neither a primitive divisor theorem for \(T_n\) nor a factorization formula for \(T_n \pm 1\).

More precisely, we shall prove the following theorem.

**Theorem 1.** There is no solution \((m, u)\) for the Brocard-Ramanujan equation (1), where \(u\) is a Tribonacci number.

The idea behind the proof is as follows. The equation we are interested in solving is \(m! = (T_n - 1)(T_n + 1)\). The 2-adic valuation of \(m!\) is \(m + O(\log m)\). We show that the 2-adic valuation of \((T_n - 1)(T_n + 1)\) is \(\leq 8 \log n / \log 2\). Thus \(m \leq 8 \log n / \log 2\). This forces \(m!\) to be smaller than \((T_n - 1)(T_n + 1)\), for \(m\) and \(n\) sufficiently large, which allows us to complete the proof.
2 The proof of Theorem 1

2.1 A key lemma

The $p$-adic order, $\nu_p(r)$, of $r$ is the exponent of the highest power of a prime $p$ which divides $r$. The $p$-adic order of a Fibonacci number was completely characterized by Lengyel [9]. Also, very recently the 2-adic order of Tribonacci numbers was made explicit by Lengyel and Marques [14]. Here, we shall prove the following key result which will play an important role in the proof of Theorem 1.

Lemma 2. We have

\[
\nu_2(T_n + 1) = \begin{cases} 
15, & \text{if } n = 61; \\
0, & \text{if } n \equiv 0, 3 \pmod{4}; \\
1, & \text{if } n \equiv 1, 2, 6 \pmod{8}; \\
3, & \text{if } n \equiv 5 \pmod{16}; \\
\nu_2((n + 3)^2) - 3, & \text{if } n \equiv 13, 29, 45 \pmod{64}; \\
\nu_2((n - 61)(n + 3)) - 3, & \text{if } n > 61 \text{ and } n \equiv 61 \pmod{64}.
\end{cases}
\]

and, for $n \geq 5$,

\[
\nu_2(T_n - 1) = \begin{cases} 
0, & \text{if } n \equiv 0, 3 \pmod{4}; \\
1, & \text{if } n \equiv 5 \pmod{8}; \\
\nu_2(n + 2) - 1, & \text{if } n \equiv 6 \pmod{8}; \\
\nu_2(n - 2) - 1, & \text{if } n \equiv 2 \pmod{8}; \\
\nu_2((n - 1)(n + 7)) - 3, & \text{if } n \equiv 1 \pmod{8}.
\end{cases}
\]

The case $T_n - 1$:

First, note that Lengyel and Marques [14] proved that $T_n - 1$ is odd for every $n \equiv 0, 3 \pmod{4}$, which proves the first case. Now, note that, in order to prove the second case, it suffices to prove that $T_n \equiv 3 \pmod{4}$. In this case, we have $n = 8k + 5$, with $k \geq 0$. Then we proceed on induction on $k$. For $k = 0$, it follows directly, since $T_5 - 1 = 7 - 1 = 6 = 2 \cdot 3$. So, we suppose that $T_{8k+5} \equiv 3 \pmod{4}$. Using the sum formula for $T_n$ (proved by Feng [16]), we have that

\[
T_{8(k+1)+5} = T_{(8k+5)+8} = T_6 T_8 T_{8k+5} + (T_6 + T_5) T_{8k+6} + T_7 T_{8k+7} = 13T_{8k+5} + 20T_{8k+6} + 24T_{8k+7} \equiv 3 \pmod{4}.
\]
In the third case, for \(t \geq 6\) and \(s \geq 1\) odd, we write \(n = 2t^3 - 3s + 2\). Now, by a Lengyel and Marques result [14, Lemma 3.1], we have that
\[
T_{2t^3 - 3s + 2} = T_{2t^3 s + 1} + T_{2t^3 - 3s} + T_{2t^3 s - 1} \\
\equiv 1 + 2t^4 + 0 \pmod{2^{t^3}} \\
\equiv 1 + 2t^4 \pmod{2^{t^3}}.
\]
This yields \(\nu_2(T_n - 1) = t - 4 = \nu_2(2^{t^3} - 3s) - 1 = \nu_2(n - 2) - 1\).

The fourth case follows by proceeding in the same way as the third one. For \(t \geq 6\) and \(s \geq 1\) odd, we write \(n = 2t^3 - 3s - 1\). Then, by the Lengyel and Marques result [14, Lemma 3.1], we have that
\[
T_{2t^3 - 3s - 1} = T_{2t^3 s + 1} - T_{2t^3 - 3s} - T_{2t^3 s - 1} \\
\equiv 1 - 2t^4 - 0 \pmod{2^{t^3}} \\
\equiv 1 - 2t^4 \pmod{2^{t^3}}.
\]
This yields \(\nu_2(T_n - 1) = t - 4 = \nu_2(2^{t^3} - 3s) - 1 = \nu_2(n + 2) - 1\).

Now, for the last case, we know that 16 divides exactly one among \(n - 1\) and \(n + 7\). Suppose that 16\(|(n + a)\), for some \(a \in \{-1, 7\}\). Then \(\nu_2(n + b) = 3\) for \(b \in \{-1, 7\} \setminus \{a\}\). So, we desire to prove that
\[\nu_2(T_n - 1) = \nu_2(n + a)\]
For that, we write \(n = 2t^3 - s - a\), for \(t \geq 6\) and \(s \geq 1\) odd, and proceed as in Lengyel and Marques [14, Lemma 3.1] to prove that
\[T_{2t^3 - s - a} - 1 \equiv 2^{t^2} \pmod{2^{t-1}}\]
Therefore
\[\nu_2(T_n - 1) = t - 2 = \nu_2(n + a) + 1,\]
and this completes the proof.

\(\square\)

The case \(T_n + 1\):

The first two cases are trivial. The third and the fourth cases follow in the same way. Note that, in order to prove them, it suffices to show that \(T_n \equiv 1 \pmod{4}\) when \(n \equiv 1, 2, 6 \pmod{8}\) and to show that \(T_n = 7 \pmod{16}\) when \(n \equiv 5 \pmod{16}\). In order to avoid unnecessary repetitions, we shall prove only one of these cases. So, let us write \(n = 8k + 6\) and apply induction on \(k \geq 0\). For \(k = 0\), it follows directly, since \(T_6 + 1 = 13 + 1 = 14 = 2 \cdot 7\). Now, suppose that \(T_{8k + 6} \equiv 1 \pmod{4}\). Then, we have that
\[
T_{8(k+1)+6} = T_{(8k+6)+8} \\
= T_6 T_{8k+6} + (T_6 + T_3) T_{8k+7} + T_7 T_{8k+8} \\
= 13 T_{8k+6} + 20 T_{8k+7} + 24 T_{8k+8} \\
\equiv 1 \pmod{4}.
\]
Now, for the fifth case, note that, if \( n = 64k + 13 \),
\[
\nu_2((n + 3)^2) - 3 = 2\nu_2(n + 3) - 3
\]
\[
= 2\nu_2(64k + 13 + 3) - 3 = 2\nu_2(16(4k + 1)) - 3 = 2 \cdot 4 - 3
\]
\[
= 5.
\]
So, it suffices to prove that \( T_n \equiv 31 \pmod{64} \). Again, we proceed on induction. First, observe that
\[
T_{13} = 927 \equiv 31 \pmod{64}.
\]
Now, we have that
\[
T_{64k+13+64} = T_{62}T_{64k+13} + (T_{62} + T_{61})T_{64k+14} + T_{63}T_{64k+15}
\]
\[
\equiv -1 + 32T_{64k+14} \pmod{64}.
\]
But, from the previous case, we have that \( T_{64k+14} \equiv 1 \pmod{4} \). Then,
\[
T_{64(k+1)+13} \equiv -1 + 32T_{64k+14} \pmod{64}
\]
\[
\equiv 32 - 1 \pmod{64}
\]
\[
\equiv 31 \pmod{64}.
\]
When \( n \equiv 29, 45 \pmod{64} \), we proceed in the same way.

For the last case, we proceed as for the last case of the previous theorem. Note that 128 divides exactly one among \( n - 61 \) and \( n + 3 \). Suppose that 128 divides \((n + a)\), for some \( a \in \{-61, 3\}\). Then \( \nu_2(n + b) = 6 \) for \( b \in \{-61, 3\} \setminus \{a\} \). So, we desire to prove that
\[
\nu_2(T_n + 1) = \nu_2(n + a) + 3.
\]
For that, we write \( n = 2^{t-2}s - a \), for \( t \geq 8 \) and \( s \geq 1 \) odd, and proceed as in Lengyel and Marques [14, Lemma 3.1] to prove that
\[
T_{2^{t-2}s-a} + 1 \equiv 2^{t+1} \pmod{2^{t+2}}.
\]
Therefore
\[
\nu_2(T_n + 1) = t + 1 = \nu_2(n + a) + 3.
\]
This completes the proof. □

2.2 The proof

If \( n \leq 61 \), a straightforward search shows that there are no solutions. So we shall suppose that \( n > 61 \). Then \( m \geq 30 \). Next we use the fact that \( \nu_2(m!) \geq m - \lfloor \log m / \log 2 \rfloor - 1 \) (which is a consequence of the De Polignac’s formula) together with Lemma 2. Then
\[
m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 1 \leq \nu_2(m!) = \nu_2(T_n - 1) + \nu_2(T_n + 1)
\]
\[
< \nu_2((n + 2)(n - 2)(n - 1)(n + 7)(n + 3)^3(n - 61)) + 5
\]
\[
\leq 8\nu_2(n + \omega) + 5,
\]
for some $\omega \in \{-61, -2, -1, 2, 3, 7\}$. Thus $\nu_2(n + \omega) \geq (m - [\log m / \log 2] - 6)/8$. Therefore, $2^{(m - [\log m / \log 2] - 6)/8} \mid (n + \omega)$. In particular, $2^{(m - [\log m / \log 2] - 6)/8} \leq |n + \omega| \leq n + 61$ (here we used that $n + \omega \neq 0$). By applying the log function, we obtain

$$\left\lfloor \frac{1}{8} \left( m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 6 \right) \right\rfloor \leq \frac{\log(n + 61)}{\log 2}. \tag{2}$$

On the other hand, $(1.83)^{2n-4} < T_n^2 = m! + 1 < 2(m/2)^m$ (the first inequality was proved by Bravo and Luca [4]). So $n < 0.9m \log(m/2) + 2.6$. Substituting this in equation (2), we obtain

$$\left\lfloor \frac{1}{8} \left( m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 6 \right) \right\rfloor \leq \frac{\log(0.9m \log(m/2) + 63.6)}{\log 2}. \tag{2}$$

This inequality yields $m \leq 78$. Then $n < 0.9 \cdot 78 \log(78/2) + 2.6 = 259.782\ldots$. Now, we use a simple routine written in Mathematica which does not return any solution in the range $30 \leq m \leq 78$ and $62 \leq n \leq 259$. The proof is complete. \qed

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\section*{References}


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