On a Conjecture on the Representation of Positive Integers as the Sum of Three Terms of the Sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$

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Abstract

We prove some cases of a conjecture by Farhi on the representation of every positive integer as the sum of three terms of the sequence $\left\lfloor \frac{n^2}{a} \right\rfloor$. This is done by generalizing a method used by Farhi in his original paper.

1 Introduction

In the following we let $\mathbb{N}$ denote the set of non-negative integers, $\lfloor \cdot \rfloor$ denote the greatest integer function, and $\langle \cdot \rangle$ denote the fractional part function.
A classical result by Legendre [3] states that every natural number not of the form $4^s(8t + 7), s, t \in \mathbb{N}$ can be written as the sum of three squares.

In relation to this, Farhi recently conjectured the following:

**Conjecture 1** (Farhi [2]). Let $a \geq 3$ be an integer. Then every natural number can be represented as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor \right)_{n \in \mathbb{N}}$.

The conjecture was confirmed by Farhi [1] and Mezroui, Azizi, and Ziane [4] for $a \in \{3, 4, 8\}$.

In this paper we generalize the method used by Farhi for $a = 4$, and partially for $a = 3$, to prove that the conjecture holds for $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}$. The method uses Legendre’s three-square theorem and properties of quadratic residues.

We also note that the set of integers $a$ such that Conjecture 1 holds is closed under multiplication by a square.

## 2 Method and results

We start by introducing the following sets:

**Definition 2.** For any nonzero $a \in \mathbb{N}$ we define

$$Q_a = \{0 < \varphi < a \mid \exists x \in \mathbb{Z}: \varphi \equiv x^2 \pmod{a}\}.$$

Therefore, $Q_a$ is the set of quadratic residues modulo $a$.

**Definition 3.** For any nonzero $a \in \mathbb{N}$ we define

$$A_a = \{\varphi \in \mathbb{N} \mid \exists x, y, z \in Q_a \cup \{0\}: \varphi = x + y + z\}.$$

Thus, $A_a$ is the set of integers that can be written as the sum of three elements of $Q_a \cup \{0\}$.

**Definition 4.** For any nonzero $a \in \mathbb{N}$ we define

$$R_a = \{\varphi \in A_a \mid \forall \psi \in A_a: \varphi \equiv \psi \pmod{a} \Rightarrow \varphi = \psi\}.$$

So, $R_a$ is the set of integers that can be written as the sum of three elements of $Q_a \cup \{0\}$, and such that no other integer in the same residue class modulo $a$ has this property.

Now we are ready to formulate the main result.

**Theorem 5.** Let $a \in \mathbb{N}$ be nonzero and assume that for every $k \in \mathbb{N}$ there exists an $r \in R_a$ such that $ak + r \neq 4^s(8t + 7)$ for any $s, t \in \mathbb{N}$. Then every $N \in \mathbb{N}$ can be written as the sum of three terms of the sequence $\left(\left\lfloor \frac{n^2}{a} \right\rfloor \right)_{n \in \mathbb{N}}$. 

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Proof. Let $N \in \mathbb{N}$ be fixed. By assumption we can choose $r \in \mathcal{R}_a$ such that $aN + r \neq 4^s(8t + 7)$ for any $s, t \in \mathbb{N}$. By Legendre’s theorem it follows that $aN + r$ can be written of the form

$$aN + r = A^2 + B^2 + C^2$$

for some $A, B, C \in \mathbb{N}$. Now we have

$$r \equiv A^2 + B^2 + C^2 \pmod{a},$$

so

$$r = (A^2 \mod{a}) + (B^2 \mod{a}) + (C^2 \mod{a}),$$

since $r \in \mathcal{R}_a$. Dividing by $a$ and separating the integer and fractional parts of the right hand side in (1), we get

$$N + \frac{r}{a} = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor + \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

and from (2) we have

$$\frac{r}{a} = \left\langle \frac{A^2}{a} \right\rangle + \left\langle \frac{B^2}{a} \right\rangle + \left\langle \frac{C^2}{a} \right\rangle,$$

so

$$N = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor.$$

Since we can find the sets $\mathcal{R}_a$ by computation, we can now apply the main theorem to get the following corollary.

**Corollary 6.** *Conjecture 1 is satisfied for $a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\}.*

*Proof.* Consider the following table:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\mathcal{R}_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>{0, 1, 2, 3}</td>
</tr>
<tr>
<td>7</td>
<td>{4, 6}</td>
</tr>
<tr>
<td>8</td>
<td>{2, 3, 5, 6}</td>
</tr>
<tr>
<td>9</td>
<td>{1, 4, 7, 8}</td>
</tr>
<tr>
<td>20</td>
<td>{11, 15, 18, 19}</td>
</tr>
<tr>
<td>24</td>
<td>{11, 14, 19, 21, 22}</td>
</tr>
<tr>
<td>40</td>
<td>{27, 38}</td>
</tr>
<tr>
<td>104</td>
<td>{99}</td>
</tr>
<tr>
<td>120</td>
<td>{107}</td>
</tr>
</tbody>
</table>
Calculating modulo 8 it can be checked fairly easily that for each \( a \in \{4, 7, 8, 9, 20, 24, 40, 104, 120\} \) and every \( k \in \mathbb{N} \) there exists an \( r \in \mathcal{R}_a \) such that \( ak + r \) is not of the form \( 4^s(8t + 7), s, t \in \mathbb{N} \), and thus every natural number can be written as the sum of three terms of the sequence \( \left\lfloor \frac{n^2}{a} \right\rfloor \) \( n \in \mathbb{N} \).

To demonstrate this, we show the case \( a = 7 \). All the other cases are done in exactly the same way.

For \( k \equiv 1, 2, 3, 6 \) or 7 (mod 8) we have \( 7k + 4 \equiv 3, 2, 1, 6 \) and 5 (mod 8), respectively, and for \( k \equiv 0, 4 \) or 5 (mod 8) we have \( 7k + 6 \equiv 6, 2 \) and 1 (mod 8), respectively. Since \( 4^s(8t + 7) \equiv 0, 4 \) or 7 (mod 8), \( s, t \in \mathbb{N} \), we conclude that for every \( k \in \mathbb{N} \) we can write \( 7k + r \), for \( r \in \mathcal{R}_7 = \{4, 6\} \), such that it is not of the form \( 4^s(8t + 7), s, t \in \mathbb{N} \). The case now follows from Theorem 5. \( \square \)

Further, one should note that the set of integers satisfying Conjecture 1 is closed under multiplication by a square.

**Observation 7.** Let \( \mathcal{M} \) be the set of integers satisfying Conjecture 1. If \( a \in \mathcal{M} \), then \( ak^2 \in \mathcal{M} \) for any integer \( k > 0 \).

**Proof.** This follows easily since for any \( n \) we can find \( A, B, C \in \mathbb{N} \) such that

\[
n = \left\lfloor \frac{A^2}{a} \right\rfloor + \left\lfloor \frac{B^2}{a} \right\rfloor + \left\lfloor \frac{C^2}{a} \right\rfloor = \left\lfloor \frac{(Ak)^2}{ak^2} \right\rfloor + \left\lfloor \frac{(Bk)^2}{ak^2} \right\rfloor + \left\lfloor \frac{(Ck)^2}{ak^2} \right\rfloor.
\]

Knowing this, we see that since Conjecture 1 is satisfied for \( a = 3, 9, 4, \) and 8, it must also hold for \( a = 3^k \) for any positive integer \( k \) and for \( a = 2^k, k > 1 \).

Finally, using Observation 7, Corollary 6, and the fact [4] that Conjecture 1 holds for \( a = 3 \), we get that the conjecture holds for the following values up to 120.

\[
a \in \{3, 4, 7, 8, 9, 12, 16, 20, 24, 27, 28, 32, 36, 40, 48, 63, 64, 72, 75, 80, 81, 96, 100, 104, 108, 112, 120\}.
\]

Unfortunately, it seems that the method deployed in Theorem 5 is not extendable to other cases, since its success relies on \( \mathcal{R}_a \), and in general \( \mathcal{R}_a \) does not contain the necessary elements for the condition in the theorem to be satisfied.

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