A Note on Catalan’s Identity for the $k$-Fibonacci Quaternions

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Abstract
Ramírez recently conjectured a version of Catalan’s identity for the $k$-Fibonacci quaternions. In this note we give a proof of (a suitably reformulated version of) this identity.

1 Introduction
For any positive real number $k$, define the $k$-Fibonacci and $k$-Lucas sequences, $(F_{k,n})_{n \in \mathbb{N}}$ and $(L_{k,n})_{n \in \mathbb{N}}$, as follows:

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad \text{and} \quad F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2,$$

and

$$L_{k,0} = 2, \quad L_{k,1} = k, \quad \text{and} \quad L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \quad n \geq 2,$$

respectively.

Let $\alpha$ and $\beta$ be the roots of the characteristic equation $x^2 - kx - 1 = 0$. Then the Binet formulas for the $k$-Fibonacci and $k$-Lucas sequences are

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$
and

$$L_{k,n} = \alpha^n + \beta^n,$$

where $\alpha = (k + \sqrt{k^2 + 4}) / 2$ and $\beta = (k - \sqrt{k^2 + 4}) / 2$.

A quaternion $p$, with real components $a_0$, $a_1$, $a_2$, $a_3$ and basis $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, is an element of the form

$$p = a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} = (a_0, a_1, a_2, a_3), \quad (a_0 \mathbf{1} = a_0),$$

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{i} \mathbf{j} = \mathbf{k} = -\mathbf{j} \mathbf{i}, \quad \mathbf{j} \mathbf{k} = \mathbf{i} = -\mathbf{k} \mathbf{j}, \quad \mathbf{k} \mathbf{i} = \mathbf{j} = -\mathbf{i} \mathbf{k}.$$

Ramírez [1] defined the $n$th $k$-Fibonacci quaternion, $D_{k,n}$, as follows:

$$D_{k,n} = F_{k,n} + F_{k,n+1} \mathbf{i} + F_{k,n+2} \mathbf{j} + F_{k,n+3} \mathbf{k}, \quad n \geq 0,$$

where $F_{k,n}$ is the $n$th $k$-Fibonacci number. Ramírez [1] also gave the Binet formula for the $k$-Fibonacci quaternion as follows:

$$D_{k,n} = \hat{\alpha} \alpha^n - \hat{\beta} \beta^n,$$

where $\hat{\alpha} = 1 + \alpha \mathbf{i} + \alpha^2 \mathbf{j} + \alpha^3 \mathbf{k}$ and $\hat{\beta} = 1 + \beta \mathbf{i} + \beta^2 \mathbf{j} + \beta^3 \mathbf{k}$.

Ramírez [1] conjectured that the Catalan identity for the $k$-Fibonacci quaternions is

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n-r} \left( 2F_{k,r}D_{k,r} - G_{k,r} \mathbf{k} \right),$$

for $n \geq r \geq 1$, where $G_{k,r}$ is the sequence satisfying the following recurrence:

$$G_{k,0} = 0, \quad G_{k,1} = k^2 + 2k, \quad \text{and} \quad G_{k,n} = (k^2 + 2) G_{k,n-1} - G_{k,n-2}, \quad n \geq 2.$$

However, in this short paper, we show that this conjecture is incorrect, by giving the correct Catalan identity and proving it.

## 2 Catalan identity for the $k$-Fibonacci quaternions

We need the following lemma.

**Lemma 1.** For $r \geq 1$, we have

$$\frac{\hat{\alpha} \hat{\beta} \beta^r - \hat{\beta} \hat{\alpha} \alpha^r}{\alpha - \beta} = -2D_{k,r} + L_{k,2}L_{k,r} \mathbf{k}.$$
Proof. Since
$$\hat{\alpha} \hat{\beta} = 2 + 2\beta i + 2\beta^2 j + (\alpha^3 + \beta^3 + \alpha - \beta) k$$
and
$$\hat{\beta} \hat{\alpha} = 2 + 2\alpha i + 2\alpha^2 j + (\alpha^3 + \beta^3 + \beta - \alpha) k,$$
we get
$$\frac{\hat{\alpha} \hat{\beta} \hat{r} - \hat{\beta} \hat{\alpha} \hat{r}}{\alpha - \beta} = -2F_{k,r} - 2F_{k,r+1} i - 2F_{k,r+2} j + (-2F_{k,r+3} + L_{k,2}L_{k,r}) k$$
$$= -2D_{k,r} + L_{k,2}L_{k,r} k.$$ 

\[ \square \]

**Theorem 2.** For \( n \geq r \geq 1 \), Catalan identity for the \( k \)-Fibonacci quaternions is
$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n+r+1} (2F_{k,r}D_{k,r} - L_{k,2}F_{k,2} k).$$

**Proof.** By considering the Binet formula for the \( k \)-Fibonacci quaternions, quaternion multiplication and Lemma 1, we obtain
$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = \left( \frac{\hat{\alpha} \alpha^{n-r} - \hat{\beta} \beta^{n-r}}{\alpha - \beta} \right) \left( \frac{\hat{\alpha} \alpha^{n+r} - \hat{\beta} \beta^{n+r}}{\alpha - \beta} \right) - \left( \frac{\hat{\alpha} \alpha^n - \hat{\beta} \beta^n}{\alpha - \beta} \right)^2$$
$$= \frac{(\alpha \beta)^n}{(\alpha - \beta)^2} \left( \frac{\hat{\alpha} \beta^{n-r} - \hat{\beta} \alpha^{n-r}}{\alpha - \beta} \right) \left( \frac{\hat{\alpha} \beta^{n+r} - \hat{\beta} \alpha^{n+r}}{\alpha - \beta} \right)$$
$$= (\alpha \beta)^n \frac{\alpha^{r} - \beta^{r}}{\alpha - \beta} \left( \frac{\hat{\alpha} \beta^r - \hat{\beta} \alpha^r}{\alpha - \beta} \right)$$
$$= (\alpha \beta)^n \frac{\alpha^{r} - \beta^{r}}{\alpha - \beta} \left( \frac{\hat{\alpha} \beta^r - \hat{\beta} \alpha^r}{\alpha - \beta} \right)$$
$$= (\alpha \beta)^n F_{k,r} (-2D_{k,r} + L_{k,2}L_{k,r} k)$$
$$= (-1)^{n-r+1} (2F_{k,r}D_{k,r} - L_{k,2}F_{k,2} k).$$

\[ \square \]

**References**

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