Fibonacci $s$-Cullen and $s$-Woodall Numbers

Diego Marques  
Departamento de Matemática  
Universidade de Brasília  
Brasília, Brazil  
diego@mat.unb.br

Ana Paula Chaves  
Instituto de Matemática e Estatística  
Universidade Federal de Goiás  
Goiás, Brazil  
apchaves@ufg.br

Abstract

The $m$-th Cullen number $C_m$ is a number of the form $m2^m + 1$ and the $m$-th Woodall number $W_m$ has the form $m2^m - 1$. In 2003, Luca and Stănică proved that the largest Fibonacci number in the Cullen sequence is $F_4 = 3$ and that $F_1 = F_2 = 1$ are the largest Fibonacci numbers in the Woodall sequence. Very recently, the second author proved that, for any given $s > 1$, the equation $F_n = ms^m \pm 1$ has only finitely many solutions, and they are effectively computable. In this note, we shall provide the explicit form of the possible solutions.

1 Introduction

A Cullen number is a number of the form $m2^m + 1$ (denoted by $C_m$), where $m$ is a nonnegative integer. This sequence was introduced in 1905 by Father J. Cullen [2] and it was mentioned in the well-known book of Guy [5, Section B20]. These numbers gained great interest in 1976, when Hooley [7] showed that almost all Cullen numbers are composite. However, despite being very scarce, it is still conjectured that there are infinitely many Cullen primes.
In a similar way, a Woodall number (also called Cullen number of the second kind) is a positive integer of the form \( m2^m - 1 \) (denoted by \( W_m \)). It is also known that almost all Woodall numbers are composite. However, it is also conjectured that the set of Woodall primes is infinite.

These numbers can be generalized to the \( s \)-Cullen and \( s \)-Woodall numbers which are numbers of the form

\[
C_{m,s} = ms^m + 1 \quad \text{and} \quad W_{m,s} = ms^m - 1,
\]

where \( m \geq 1 \) and \( s \geq 2 \). This family was introduced by Dubner [3]. A prime of the form \( C_{139948,151} \) is \( 3 \times 2^{10^5} + 1 \) an integer with 304949 digits.

Many authors have searched for special properties of Cullen and Woodall numbers and their generalizations. We refer the reader to [4, 6, 9, 10] for classical and recent results on this subject.

In 2003, Luca and Stănică [8, Theorem 3] proved that the largest Fibonacci number in the Cullen sequence is \( F_4 = 3 = 1 \cdot 2^1 + 1 \) and that \( F_1 = F_2 = 1 = 1 \cdot 2^1 - 1 \) are the largest Fibonacci numbers in the Woodall sequence.

Recall that \( \nu_p(r) \) denotes the \( p \)-adic order of \( r \), which is the exponent of the highest power of a prime \( p \) which divides \( r \). Also, the order (or rank) of appearance of \( n \) in the Fibonacci sequence, denoted by \( z(n) \), is defined as the smallest positive integer \( k \), such that \( n \mid F_k \) (for results on this function, see [13] and references therein). Let \( p \) be a prime number and set \( e(p) := \nu_p(F_{z(p)}) \).

Very recently, Marques [11] proved that if the equation

\[
F_n = ms^m + \ell
\]

has solution, with \( m > 1 \) and \( \ell \in \{\pm 1\} \), then \( m < (6.2 + 1.9e(p)) \log(3.1 + e(p)) \), for some prime factor \( p \) of \( s \). This together with the fact that \( e(p) = 1 \) for all prime \( p < 2.8 \cdot 10^{16} \) (PrimeGrid, March 2014) implies that there is no Fibonacci number that is also a nontrivial (i.e., \( m > 1 \)) \( s \)-Cullen number or \( s \)-Woodall number when the set of prime divisors of \( s \) is contained in \( \{2, 3, 5, \ldots, 2799999999999991\} \). This is the set of the first 759997990476073 prime numbers.

In particular, the previous result ensures that for any given \( s \geq 2 \), there are only finitely many Fibonacci numbers which are also \( s \)-Cullen numbers or \( s \)-Woodall numbers and they are effectively computable.

In this note, we shall invoke the primitive divisor theorem to provide explicitly the possible values of \( m \) satisfying Eq. (1). More precisely,

**Theorem 1.** Let \( s > 1 \) be an integer. Let \((n, m, \ell)\) be a solution of the Diophantine equation (1) with \( n, m > 1 \) and \( \ell \in \{-1, 1\} \). Then \( m = e(p)/\nu_p(s) \), for some prime factor \( p \) of \( s \).

In particular, we have that \( m \leq e(p) \) for some prime factor \( p \) of \( s \). Also, we can deduce [11, Corollary 3] from the above theorem. In fact, for all \( p < 2.8 \cdot 10^{16} \) we have \( e(p) = 1 \) and then if \((n, m, \ell)\) is a solution, with \( m > 1 \), we would have the contradiction that \( 1 < m = e(p)/\nu_p(s) = 1/\nu_p(s) \) for some \( p \) dividing \( s \).
2 The proof

Suppose that \( n \leq 27 \). Then \( \max\{2s^2 - 1, m_2^m - 1\} \leq ms^m + \ell = F_n \leq F_{27} = 196418 \) yields \( s \leq 313 \) and \( m \leq 13 \). For this, we prepare a simple Mathematica program which, in a few seconds, does not return any solution with \( m > 1 \).

So we may suppose that \( n \geq 28 \). We rewrite Eq. (1) as \( F_n - \ell = ms^m \). It is well-known that \( F_n \pm 1 = F_aL_b \), where \( 2a, 2b \in \{n \pm 2, n \pm 1\} \). (This factorization depends on the class of \( n \) modulo 4. See [12, (3)] for more details.) Then the main equation becomes

\[
F_aL_b = ms^m,
\]

where \( 2a, 2b \in \{n \pm 2, n \pm 1\} \) and \(|a - b| \in \{1, 2\} \). Since \(|a - b| \in \{1, 2\} \), then \( \gcd(a, b) \in \{1, 2\} \) and then \( \gcd(F_a, L_b) = 1, 2 \) or 3. Therefore, we have \( F_a = m_1s_1 \) and \( L_b = m_2s_2 \), where \( m_1m_2 = m, s_1s_2 = s \) and \( \gcd(m_1, m_2), \gcd(s_1, s_2) \in \{1, 2, 3\} \). We claim that \( s_1 > 1 \).

Suppose, to get a contradiction, that \( s_1 = 1 \), then \( F_a = m_1 \) and \( L_b = m_2s^m \). Since \( 2a - 4 \geq n - 6 \geq (n + 8)/2 \geq b + 3 \), we arrive at the following contradiction:

\[
m^2 \geq m_1^2 = F_a^2 \geq \alpha^{2a-4} \geq \alpha^{b+3} \geq 2L_b = 2m_2s^m \geq 2^{m+1} > m^2,
\]

where \( \alpha = (1 + \sqrt{5})/2 \). Here, we used that \( F_j \geq \alpha^{j-2} \) and \( L_j \leq \alpha^{j+1} \). Thus \( s_1 > 1 \). Since \( a \geq (n - 2)/2 \geq 13 \), then by the primitive divisor theorem (see [1]), there exists a primitive divisor \( p \) of \( F_a \) (i.e., \( p \mid F_a \) and \( p \nmid F_1 \cdots F_{a-1} \)). We also have that \( p \equiv \pm 1 \pmod{a} \). In particular, \( p \geq a - 1 \). Thus \( p \mid F_a = m_1s_1^m \). Suppose that \( p \mid m_1 \). In this case, one has that \( a - 1 \leq p \leq m_1 \leq m \). On the other hand, we get

\[
2^m \leq m_1s_1^m = F_a \leq \alpha^{a-1} < 2^{a-1}.
\]

Thus \( m < a - 1 \) which gives a contradiction. Therefore \( p \nmid m_1 \) and consequently \( p \mid s_1 \). This yields \( \nu_p(F_a) = m\nu_p(s_1) = m\nu_p(s) \) (because \( p > 3, s = s_1s_2 \) and \( \gcd(s_1, s_2) \leq 3 \)). On the other hand, \( z(p) = a \) and so \( \nu_p(F_{z(p)}) = \nu_p(F_a) = m\nu_p(s) \) as desired.

3 Acknowledgements

The first author is grateful to FAP-DF and CNPq for financial support. The authors wish to thank the editor and the referee for their helpful comments.

References

[1] R. D. Carmichael, On the numerical factors of the arithmetic forms \( \alpha^n \pm \beta^n \). Ann. of Math. 15 (1913), 30–70.


2010 Mathematics Subject Classification: Primary 11B39.

Keywords: Fibonacci number, Cullen number.

(Concerned with sequences A000045 and A002064.)

Received October 11 2014; revised version received December 24 2015. Published in *Journal of Integer Sequences*, January 6 2015.

Return to *Journal of Integer Sequences* home page.