A Generating Function for the Diagonal $T_{2n,n}$ in Triangles

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Abstract
We present techniques for obtaining a generating function for the diagonal $T_{2n,n}$ of the triangle formed from the coefficients of a generating function $G(x)$ raised to the power $k$. We obtain some relations between central coefficients and coefficients of the diagonal $T_{2n,n}$, and we also give some examples.

1 Introduction
A triangle is a classic object of research in combinatorics. For instance, the Pascal triangle, the Bernoulli-Euler triangle, the Catalan triangle, and the Motzkin triangle are discussed in many papers and books [3, 10, 7].
Let $G(x)$ be an ordinary power series without a constant term, i.e., $G(x) = \sum_{n>0} g_n x^n$, where $g_0 = 0$ and $g_1 \neq 0$. In this paper we deal with the triangle $T_{n,k}$ defined as follows:

$$[G(x)]^k = \sum_{n\geq k} T_{n,k}x^n.$$ 

Here we assume that $G(x)^0 = T_{0,0} = 1$.

Then the generating function $G(x)$ raised to the power $k$ gives the following triangle $T_{n,k}$

$$
\begin{array}{cccccc}
T_{1,1} & & & & & \\
T_{2,1} & T_{2,2} & & & & \\
T_{3,1} & T_{3,2} & T_{3,3} & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
T_{n,1} & T_{n,2} & \cdots & \cdots & T_{n,n-1} & T_{n,n}
\end{array}
$$

The following notation will be used throughout this paper. The authors [4, 5] introduced the notion of the *composita* of a given ordinary generating function $G(x) = \sum_{n>0} g(n)x^n$.

**Definition 1.** The composita is the function of two variables defined by

$$G^\Delta(n,k) = \sum_{\pi_k \in C_n} g(\lambda_1)g(\lambda_2)\cdots g(\lambda_k),$$

(1)

where $n, k, \lambda_i$ are integers that are greater than 0, $C_n$ is the set of all compositions of $n$, and $\pi_k$ is the composition into $k$ parts exactly ($\sum_{i=1}^k \lambda_i = n$).

The generating function of the composita is equal to

$$[G(x)]^k = \sum_{n \geq k} G^\Delta(n,k)x^n = \sum_{n \geq k} T_{n,k}x^n.$$  

(2)

This notation coincides with the concept of Riordan array \((1,G(x))\) or \(\left(\frac{G(x)}{x},G(x)\right)\), which was given by Shapiro, Getu, Woan, and Woodson [8].

Recently, in [6], we have shown how to find a generating function of the central elements of such triangles

$$C(x) = \sum_{n>0} T_{2n-1,n}x^{n-1} = F'(x),$$

(3)

where $F(x)$ is the solution of the equation

$$F(x) = xS(F(x))$$

(4)

and

$$xS(x) = G(x).$$

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For solving (4), one uses the Lagrange inversion formula (LIF), which was proved by Stanley [10]. In [6], we applied the LIF for the generating functions raised to the $k$ power:

$$[G(x)]^k = \sum_{n \geq k} G^\Delta(n, k) x^n$$

and

$$[F(x)]^k = \sum_{n \geq k} F^\Delta(n, k) x^n.$$ 

We obtained the following relation between two triangles:

$$F^\Delta(n, k) = \frac{k}{n} T_{2n-k,n}.$$ 

In this paper we present a method for obtaining the generating function for the diagonal $T_{2n,n}$ of a triangle $T_{n,k}$. The triangle is given by the following expression

$$[G(x)]^k = \sum_{n \geq k} T_{n,k} x^n.$$ 

2 Main results

The main result of this paper is given in the following theorem.

Theorem 2. Suppose we have the generating function $G(x) = \sum_{n>0} g_n x^n$ that forms a triangle $T_{n,k}$:

$$[G(x)]^k = \sum_{n \geq k} T_{n,k} x^n.$$ 

Then the generating function $A(x) = \sum_{n \geq 0} T_{2n,n} x^n$ for the diagonal $T_{2n,n}$ of the triangle is defined by

$$A(x) = \frac{x F'(x)}{F(x)},$$

where $F(x) = x S(F(x))$ with $S(x) = \frac{G(x)}{x}$.

Proof. Suppose we have the following Laurent series

$$\Phi(z) = \varphi z + \varphi_0 + \frac{\varphi_1}{z} + \cdots + \frac{\varphi_n}{z^n} + \cdots$$

Then, raising this generating function to the power $k$, we get

$$[\Phi(z)]^k = \Phi_k(z) + E_k(z),$$

where $\Phi_k(z)$ contains the nonnegative powers of $z$ and $E_k(z)$ contains the remaining powers of $z$. According to Suetin [11], $\Phi_k(z)$ is the Faber polynomial.
Let us consider the generating function \( G(z) \) in terms of \( \Phi(z) \). That is,

\[
[G(z)]^k = [z^2 \Phi(1/z)]^k = \sum_{n \geq k} T_{n,k} z^n.
\]

Then we have

\[
[\Phi(z)]^k = z^{2k} \sum_{n \geq k} T_{n,k} z^{-n}.
\]

After transformation, the Faber polynomial is equal to

\[
\Phi_n(z) = \sum_{k=0}^n T_{2n-k,n} z^k,
\]

(6)

For the case \( z = 0 \), we have

\[
\Phi_n(0) = T_{2n,n}.
\]

(7)

According to Curtiss [2] and Suetin [11], the generating function for the Faber polynomials is equal to

\[
\frac{t\phi'(t)}{\phi(t)} - z = \sum_{n \geq 0} \Phi_n(z) t^{-n},
\]

where \( \phi(t) \) is the compositional inverse of \( \Phi(t) \).

Then the generating function for the case \( z = 0 \) is equal to

\[
\frac{t\phi'(t)}{\phi(t)} = \sum_{n \geq 0} \Phi_n(0) t^{-n}.
\]

Next we set \( t = \frac{1}{x} \). Taking into account that

\[
(\phi(1/x))' = \phi'(1/x) (1/x)' = \frac{-\phi'(1/x)}{x^2}
\]

or

\[
\phi'(1/x) = -x^2 (\phi(1/x))'
\]

we get the generating function for \( \Phi_n(0) \)

\[
A(x) = -\frac{x(\phi(1/x))'}{\phi(1/x)} = \sum_{n \geq 0} \Phi_n(0) x^n.
\]

(8)

Since \( \phi(t) \) is the compositional inverse of \( \Phi(t) \), the following identity holds:

\[
\Phi(\phi(t)) = t.
\]

If we substitute \( 1/x \) for \( t \), then we obtain the following relation:

\[
\Phi(\phi(1/x)) = 1/x.
\]
Since 
\[ \Phi(x) = x^2G(1/x) = xS(1/x), \]
we get 
\[ \phi(1/x)S(1/\phi(1/x)) = \frac{1}{x}. \]
Then 
\[ \frac{1}{\phi(1/x)} = xS(1/\phi(1/x)). \]
According to (4), we have 
\[ \frac{1}{\phi(1/x)} = F(x) \]
Therefore, according to (7) and (8), the generating function for the diagonal \( T_{2n,n} \) is equal to 
\[ A(x) = -\frac{x}{F(x)} = \sum_{n \geq 0} T_{2n,n} x^n. \]
The theorem is thus proved.

As applications of Theorem 2, we give the following examples.

**Example 3.** Let us consider the Pascal triangle. This triangle can be defined by the generating function 
\[ [G(x)]^k = \left( \frac{x}{1-x} \right)^k = \sum_{n \geq k} \binom{n-1}{n-k} x^n \]
The solution of the equation 
\[ F(x) = \frac{x}{1-F(x)} \]
is the generating function 
\[ F(x) = \frac{1 - \sqrt{1-4x}}{2}. \]
Therefore, the generating function for the diagonal \( T_{2n,n} \) with the general term \( \binom{2n-1}{n} \) is 
\[ A(x) = \frac{x}{F(x)} = \frac{2x}{(1-\sqrt{1-4x}) \sqrt{1-4x}}. \]

**Example 4.** Let us find the generating function \( A(x) = \sum_{n \geq 0} T_{2n,n} x^n \) for the triangle defined by the following generating function 
\[ G(x) = x + x^2 + x^3. \]
Solving the equation 
\[ F(x) = x(1 + F(x) + F(x)^2), \]
we get the generating function for the Motzkin numbers (see the sequence A001006 in [9])

\[ F(x) = \frac{-\sqrt{-3x^2 - 2x + 1} - x + 1}{2x}. \]

Then we have

\[ \frac{x F'(x)}{F(x)} = \frac{\sqrt{-3x^2 - 2x + 1} + x - 1}{(x - 1) \sqrt{-3x^2 - 2x + 1} - 3x^2 - 2x + 1}. \]

After transformation, we obtain

\[ A(x) = \frac{1}{\sqrt{-3x^2 - 2x + 1}}. \]

**Example 5.** Let us find the generating function \( A(x) = \sum_{n \geq 0} T_{2n,n} x^n \) for the triangle defined by the following expression

\[ [G(x)]^k = \left[ \frac{1 - \sqrt{1 - 4x}}{2} \right]^k = \sum_{n \geq k} \frac{k(2n-k-1)}{n} x^n. \]

The solution of the functional equation (4) for this case is the following generating function (see sequence A001764 in [9])

\[ F(x) = \frac{2}{\sqrt{3x}} \sin \left( \frac{1}{3} \arcsin \left( \frac{\sqrt{27x}}{2} \right) \right). \]

Therefore, the desired generating function has the form

\[ A(x) = \frac{x F'(x)}{F(x)} = 1 + \sum_{n > 0} \frac{(3n-1)}{n} \frac{x^n}{2} = \frac{\sqrt{3x}}{2\sqrt{4 - 27x}} \cot \left( \frac{1}{3} \arcsin \left( \frac{\sqrt{27x}}{2} \right) \right) + \frac{1}{2}. \]

**Example 6.** Let us consider the triangle defined by the expression

\[ [G(x)]^m = [x^2 \cot(x)]^m = \sum_{n \geq m} T_{n,m} x^n, \]

where

\[ T_{n,m} = (-1)^{\frac{n-m}{2}} \sum_{l=0}^{m} \frac{2^{n-2m+l} (m)}{(n-2m+l)!} \sum_{k=0}^{n-2m+l} s(l+k, l) S(n-2m+l, k) \left( \begin{array}{c} k+l \nonumber \end{array} \right). \]

Here \( s(n, k) \) and \( S(n, k) \) stand for the Stirling numbers of the first and second kinds, respectively [1, 3].
This triangle forms the sequence A199542 in [9]. Then we have

\[ T_{2n,n} = (-1)^{\frac{n}{2}} \sum_{l=0}^{n} 2^l \left( \sum_{k=0}^{l} \frac{k! S(l, k) s(l + k, l)}{(l + k)!} \right) \binom{n}{l}. \]

For the equation \( F(x) = x F(x) \cot(F(x)) \), the solution is the generating function \( \arctan(x) \).

Hence,

\[ \frac{x F'(x)}{F(x)} = \frac{x}{(1 + x^2) \arctan(x)} = 1 - \frac{2x^2}{3} + \frac{26x^4}{45} - \frac{502x^6}{945} + \frac{7102x^8}{14175} + \cdots \]

Therefore, we obtain

\[ A(x) = \frac{x}{(1 + x^2) \arctan(x)} = \sum_{n \geq 0} (-1)^{\frac{n}{2}} \sum_{l=0}^{n} 2^l \left( \sum_{k=0}^{l} \frac{k! S(l, k) s(l + k, l)}{(l + k)!} \right) \binom{n}{l} x^n. \]

Next, we derive some interesting identities between coefficients in triangles.

**Theorem 7.** Suppose we have the triangle \( T_{n,k} \), which is generated by \( G(x)^k = \sum_{n \geq k} T_{n,k} x^n \).

Then the following identity holds for the central coefficients of the triangle

\[ T_{2n-1,n} = \sum_{i=1}^{n} \frac{1}{i} T_{2i-1,i} T_{2(n-i),n-i}. \]  

**Proof.** The result follows from Theorem 2 and the expression (3). We point out that

\[ F(x) = \sum_{n \geq 0} \frac{1}{n} T_{2n-1,n} x^n \]

and

\[ \frac{x F'(x)}{F(x)} = \sum_{n \geq 0} T_{2n,n} x^n. \]

Since

\[ x F'(x) = \left( \frac{x F'(x)}{F(x)} \right) F(x), \]

by applying the multiplication rule for formal power series, we obtain the desired result. □

**Example 8.** Using Theorem 7, we obtain the identities for the Stirling numbers.

The Stirling numbers of the first kind \( s(n,k) \) count the number of permutations of \( n \) elements with \( k \) disjoint cycles. The Stirling numbers of the first kind are defined by the following generating function [1]:

\[ \psi_k(x) = \sum_{n \geq k} s(n,k) \frac{x^n}{n!} = \frac{1}{k!} \ln^k(1 + x). \]
With the help of (9), we find the following identity for the Stirling numbers of the first kind:

\[ s(2n - 1, n) = \sum_{i=1}^{n} \frac{(2n-1)}{(n-1)} \binom{n}{i} s(2i - 1, i) s(2(n - i), n - i). \]

The Stirling numbers of the second kind \( S(n, k) \) count the number of ways to partition a set of \( n \) elements into \( k \) nonempty subsets. The Stirling numbers of the second kind are defined by the following generating function [1]:

\[ \Phi_k(x) = \sum_{n \geq k} S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k. \]

Using Theorem 7, we derive the following identity

\[ S(2n - 1, n) = \sum_{i=1}^{n} \frac{(2n-1)}{(n-1)} \binom{n}{i} S(2i - 1, i) S(2(n - i), n - i). \]

**Example 9.** Suppose we have the triangle defined by the following expression

\[ (xe^x)^k = \sum_{n \geq k} \frac{k^{n-k}}{(n-k)!} x^n. \]

Then, using (9), we get

\[ \frac{n^{n-1}}{(n-1)!} = \sum_{i=1}^{n} \frac{i^{i-2}}{(i-1)!} \frac{(n-i)^{n-i}}{(n-i)!} \]

or after simple manipulation

\[ n^{n-1} = \sum_{i=1}^{n} \binom{n-1}{i-1} i^{i-2} (n-i)^{n-i}. \]

**Example 10.** Suppose we have the triangle defined by the following expression

\[ \left( \frac{x}{(1-x)^m} \right)^k = \sum_{n \geq k} \binom{n + (m-1)k - 1}{n-k} x^n. \]

Then, according to (9), we obtain

\[ \binom{(m+1)n-2}{n-1} = \sum_{i=1}^{n} \frac{1}{i} \binom{im + i - 2}{i-1} \binom{(m+1)n - im - i - 1}{n-i}. \]

If we put \( m = 2 \), we derive the following identity

\[ \binom{3n-2}{n-1} = \sum_{i=1}^{n} \frac{3i-2}{i-1} \binom{3n-3i-1}{n-i}. \]
Example 11. Suppose we have the triangle defined by the following expression

\[ [G(x)]^k = \left( \frac{x^2}{e^x - 1} \right)^k = \sum_{n \geq k} T_{n,k} x^n, \]

where

\[ T_{n,m} = \frac{m!}{(n-m)!} \sum_{k=0}^{n-m} \frac{k! S_1(m+k,m) S_2(n-m,k)}{(m+k)!}. \]

The solution of the equation (4) for this case, that is, \( F(x) = x \frac{F(x)}{e^{F(x)} - 1} \), is the generating function \( \ln(1 + x) \) (see the sequence A191578 in [9]).

Then, according to Theorem 2, we have

\[ \frac{x F'(x)}{F(x)} = \frac{x}{(1+x) \ln(1 + x)} = \sum_{n \geq 0} T_{2n,n} x^n. \]

where

\[ T_{2n,n} = \sum_{k=0}^{n} \frac{k! S_2(n,k) S_1(n+k,n)}{(n+k)!}. \]

This is reflected in the sequence A002208 in [9].

Using Theorem 7, we obtain the following identity

\[ \sum_{m=0}^{n-1} (-1)^m \sum_{k=0}^{m} \frac{k! S_2(m,k) S_1(m+k,m)}{(m+k)!} = 1. \]

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References


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