A Note on the Generating Function for the
Stirling Numbers of the First Kind

Ricky X. F. Chen
Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55
DK-5230, Odense M
Denmark
chen.ricky1982@gmail.com

Abstract
In this short note, we present a simple constructive proof for the generating function
for the unsigned Stirling numbers of the first kind using the equidistribution of pilots
and cycles of permutations.

1 Introduction
There are many studies on different statistics of permutations in the literature, e.g., inver-
sion number, excedance and descent [4]. In this note, we study another simple statistic of
permutations which we call pilots (while they could be called right-to-left minima as well).
For a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \) on \([n] = \{1, 2, \ldots, n\}\), \(\pi_i\) is called a pilot of \(\pi\) if \(\pi_i < \pi_j\)
for all \(j > i\). Note that \(\pi_n\) is always a pilot of \(\pi\). We relate pilots to a representation of a
permutation as a product of its disjoint cycles, that allows us to give a simple constructive
proof for the generating function for the unsigned Stirling numbers of the first kind.

The unsigned Stirling number of the first kind \(c(n, k)\) (see A132393 [3]) is the number
of permutations on \([n]\) consisting of \(k\) disjoint cycles [2, 4]. Our main result is to prove the
following theorem:
Theorem 1. For $1 \leq k \leq n$, we have
\[
\sum_{k=1}^{n} c(n, k)x^k = x(x+1)(x+2) \cdots (x+n-1).
\] (1)

2 Proof of Theorem 1

There are four proofs of Eq. (1) in Stanley [4] and one in Callan [1]. In Stanley [4], when a permutation $\pi$ is written as product of its disjoint cycles, a standard representation is defined as follows: each cycle is written with its largest element first, and all the cycles are written in increasing order of their largest element. By this standard representation, we can obtain a bijection between permutations with $k$ cycles and permutations with $k$ left-to-right maxima. However, to make use of pilots, we define a different representation as follows: we write $\pi = C_1C_2 \cdots C_k$ so that $\min\{C_i\} < \min\{C_j\}$ for all $j > i$ and each cycle $C_i$ ends with $\min\{C_i\}$ for all $i$. We call this new representation as the standard representation of type $P$.

For example, $\pi = 76154832$ has three cycles: $(1,7,3)$, $(2,6,8)$ and $(4,5)$. Then, in the standard representation of type $P$, we write $\pi = (731)(682)(54)$.

For a permutation $\pi$ with $k$ cycles written in the standard representation of type $P$, if we erase the parentheses of the cycles, we obtain a permutation as a word $\pi'$. For example, from $\pi = (731)(682)(54)$ we obtain $\pi' = 73168254$. Reversely, each pilot of $\pi'$ induces a cycle of $\pi$, e.g., $1 \rightarrow (731)$, $2 \rightarrow (682)$, $4 \rightarrow (54)$. It is easy to observe that such a correspondence between permutations with $k$ cycles and permutations with $k$ pilots is a bijection, that is, we have

Lemma 2. The number of permutations with $k$ pilots equals to the number of permutations with $k$ cycles.

Let $\text{pil}(\pi)$ denote the number of pilots of $\pi$. Our idea to prove Eq. (1) is to show that
\[
\sum_{\pi} x^{\text{pil}(\pi)} = x(x+1)(x+2) \cdots (x+n-1),
\]
where the sum is over all permutations $\pi$ on $[n]$.

Proof of Theorem 1. Note that $\pi_1$ is a pilot of $\pi = \pi_1\pi_2 \cdots \pi_n$ if and only if $\pi_1 = 1$; the other $n-1$ cases will not make $\pi_1$ a pilot. The element $\pi_2$ is a pilot of $\pi$ if and only if $\pi_2 = \min\{[n] \setminus \{\pi_1\}\}$; the remaining $n-2$ cases, i.e., $\pi_2 \in [n] \setminus \{\pi_1, \min\{[n] \setminus \{\pi_1\}\}\}$, will not make $\pi_2$ a pilot; and so on and so forth. In summary, to construct a permutation $\pi$ starting from an empty word, suppose $\pi_j$ has been determined for $1 \leq j \leq i-1$, then $\pi_i$ has only one chance to be a pilot of $\pi$, i.e., $\pi_i = \min\{[n] \setminus \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}\}$, and the other $n-i$ cases
not. Hence,

\[ \sum_{\pi} x^{\text{pil}(\pi)} = (x + n - 1) \times (x + n - 2) \cdots \times (x + 1) \times x \]

by \( \pi_1 \) \( \pi_2 \) \( \vdots \) \( \pi_{n-1} \) \( \pi_n \).

Therefore, Eq. (1) holds from Lemma 2. \( \square \)

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References


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(Concerned with sequence A132393.)

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