Pattern Popularity in Multiply Restricted Permutations

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Abstract
We derive explicit formulae or generating functions for the popularity of all the length-3 patterns in multiply restricted permutations, and provide combinatorial interpretations for some non-trivial equipopular patterns as well.

1 Introduction
Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) be a permutation in the symmetric group \( S_n \). We say that \( \sigma \) contains a pattern \( q = q_1 q_2 \cdots q_k \in S_k \) if there exist \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that the entries \( \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) have the same relative order as the entries of \( q \), i.e., \( q_j < q_l \) if and only if \( \sigma_{i_j} < \sigma_{i_l} \) whenever \( 1 \leq j, l \leq k \). We say that \( \sigma \) avoids \( q \) if \( \sigma \) does not contain \( q \) as a pattern. A permutation may contain multiple copies of a pattern. For example, permutation 43512 contains two copies of pattern 321, namely 431 and 432, but avoids pattern 123.

For a pattern \( q \), let \( S_n(q) \) denote the set of all permutations in \( S_n \) that avoid the pattern \( q \), and for \( R \subseteq S_k \), let \( S_n(R) = \bigcap_{q \in R} S_n(q) \) be the set of permutations in \( S_n \) that avoid every pattern contained in \( R \). For two permutations \( \sigma \) and \( q \), we set \( f_q(\sigma) \) to be the number of copies of \( q \) in \( \sigma \) as a pattern. The popularity of pattern \( q \) in \( S_n(R) \) is defined as

\[
f_q(S_n(R)) = \sum_{\sigma \in S_n(R)} f_q(\sigma).
\]
We say that \( p \) and \( q \) are equipopular if \( f_p(S_n(R)) = f_q(S_n(R)) \) for all \( n \).

The complement of \( \sigma \) is given by \( \sigma^c = (n + 1 - \sigma_1)(n + 1 - \sigma_2) \cdots (n + 1 - \sigma_n) \), its reverse is defined as \( \sigma^r = \sigma_n \cdots \sigma_2 \sigma_1 \) and the inverse \( \sigma^{-1} \) is the regular group-theoretic inverse permutation. For any set of permutations \( R \), let \( R^c \) be the set obtained by complementing each element of \( R \), and the sets \( R^r \) and \( R^{-1} \) are defined analogously. It is well known that

**Lemma 1.** Let \( R \subseteq S_k \) be any set of permutations in \( S_k \), and \( \sigma \in S_n \), we have

\[
\sigma \in S_n(R) \iff \sigma^c \in S_n(R^c) \iff \sigma^r \in S_n(R^r) \iff \sigma^{-1} \in S_n(R^{-1}).
\]

Cooper [6] first raised the problem of determining the total number \( f_q(S_n(r)) \), and Bóna [2] derived the generating function of the sequence \( (f_q(S_n(132)))_{n \geq 1} \) for monotone pattern, i.e., \( q = 12 \cdots k \) or \( q = k(k - 1) \cdots 21 \). Further, Bóna [3] studied the generating functions for other length-3 patterns in \( S_n(132) \), and showed both algebraically and bijectively that

\[
f_{231}(S_n(132)) = f_{312}(S_n(132)) = f_{213}(S_n(132)).
\]

According to the correspondence between 132-avoiding permutations and binary plane trees, Rudolph [13] showed that patterns of equal length are equipopular if their associated binary plane trees have identical spine structure. For the converse direction, Chua and Sankar [4] gave a complete classification of 132-avoiding permutations into equipopularity classes. Moreover, Homberger [9] presented exact formulae for the occurrences of each length-3 pattern in \( S_n(123) \). From Lemma 1 and the existing results on \( S_n(123) \) and \( S_n(132) \), we can obtain the popularity of each length-3 pattern for the singly restricted permutations \( S_n(r) \) with \( r = 213, 231, 312, 321 \). Therefore, it is well-studied for the popularity of length-3 patterns in singly restricted permutations, whereas it remains open for multiply restricted permutations.

In this paper, we focus on counting the number of occurrences of length-3 patterns in multiply restricted permutations \( S_n(R) \) for \( R \subset S_3 \), especially for double and triple restrictions.
We obtain exact formulae or generating functions for popularity of each length-3 pattern, and the detailed results are summarized in Table 1. Moreover, we present combinatorial proofs for non-trivial equalities between the number of occurrences of different patterns. It is routine to consider the restricted permutations of higher multiplicity since there are only finite permutations, as shown in [14, Proposition 17]. Therefore, this work gives a complete study on the popularity of length-3 patterns in the multiply restricted permutations. For the distributions of other statistics in multiply restricted permutations, see [7, 8, 10, 11, 12].

2 Doubly restricted permutations

This section deals with the enumeration of the popularity for length-3 patterns in the doubly restricted permutations, i.e., permutations avoiding two different patterns in \( S_3 \). For doubly restricted permutations, we have the following proposition from [14].

**Proposition 2.** ([14, Lemma 5]) For every symmetric group \( S_n \),

1. \(|S_n(123, 132)| = |S_n(123, 213)| = |S_n(231, 321)| = |S_n(312, 321)| = 2^{n-1};
2. \(|S_n(132, 213)| = |S_n(231, 312)| = 2^{n-1};
3. \(|S_n(132, 231)| = |S_n(213, 312)| = 2^{n-1};
4. \(|S_n(132, 312)| = |S_n(213, 321)| = 2^{n-1};
5. \(|S_n(132, 321)| = |S_n(123, 231)| = |S_n(123, 312)| = |S_n(213, 321)| = \binom{n}{2} + 1;
6. \(|S_n(123, 321)| = 0 \text{ for } n \geq 5.

Thus it is sufficient to consider the pattern popularity for the first set from class 1 to class 5, and the pattern popularity for the other sets can be derived by taking complement, reverse or inverse.

A composition of \( n \) is an expression of \( n \) as an ordered sum of positive integers, and we say that \( c \) has \( k \) parts or \( c \) is a \( k \)-composition if there are exactly \( k \) summands appeared in composition \( c \). Let \( C_n \) and \( C_{n,k} \) denote the set of all compositions of \( n \) and the set of \( k \)-compositions of \( n \), respectively. It is known that \(|C_0| = 1\), and for \( n \geq 1 \), \( 1 \leq k \leq n \), \(|C_n| = 2^{n-1}\) and \(|C_{n,k}| = \binom{n-1}{k-1}\). For more details on compositions, see [16]. It is helpful to introduce a lemma as follows:
Lemma 3. For $n \geq 1$, we have

$$a(n) := \sum_{c_1 + c_2 + \cdots + c_k = n} c_k = 2^n - 1,$$

$$b(n) := \sum_{c_1 + c_2 + \cdots + c_k = n} c_k(c_k - 1) = 2^{n+1} - 2n - 2,$$

$$c(n) := \sum_{c_1 + c_2 + \cdots + c_k = n} k = (n + 1)2^{n-2},$$

$$d(n) := \sum_{c_1 + c_2 + \cdots + c_k = n} k(k - 1) = (n^2 + n - 2)2^{n-3},$$

where the sums are taken over all compositions of $n$.

Proof. For $c_k = m$, we can regard $c_1 + c_2 + \cdots + c_{k-1}$ as a composition of $n - m$. Since the number of compositions of $n - m$ is $2^{n-m-1}$ for $1 \leq m \leq n - 1$ and the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$, we have

$$a(n) = n + \sum_{m=1}^{n-1} m2^{n-m-1}, \quad b(n) = n(n-1) + \sum_{m=1}^{n-1} m(m-1)2^{n-m-1},$$

and

$$c(n) = \sum_{k=1}^{n} k\binom{n-1}{k-1}, \quad d(n) = \sum_{k=1}^{n} k(k-1)\binom{n-1}{k-1}.$$

Let $g(x) = \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$ and $h(x) = x \sum_{i=1}^{n} \binom{n-1}{i-1} x^{i-1} = x(1+x)^{n-1}$. We have

$$g'(x) = \sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2},$$

$$g''(x) = \sum_{i=1}^{n-1} i(i-1)x^{i-2} = \frac{(3n - 2n^2 - 2)x^n + (2n^2 - 4n)x^{n-1} + (n - n^2)x^{n-2} + 2}{(1-x)^3},$$

$$h'(x) = \sum_{i=1}^{n} i\binom{n-1}{i-1} x^{i-1} = (nx + 1)(1+x)^{n-2},$$

$$h''(x) = \sum_{i=1}^{n} i(i-1)\binom{n-1}{i-1} x^{i-2} = [n^2x + n(2-x) - 2] (1+x)^{n-3}.$$

It follows that

$$a(n) = 2^{n-2}g'(1/2) + n, \quad b(n) = 2^{n-3}g''(1/2) + n(n-1),$$

and

$$c(n) = h'(1), \quad d(n) = h''(1).$$

Lemma 3 holds by simple computations.
2.1 Pattern popularity in $(123,132)$-avoiding permutations

In this subsection, we calculate the popularity of all length-3 patterns in $S_n(123,132)$. For a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, $\sigma_i$ is said to be a left-to-right maximum (resp., right-to-left maximum) if $\sigma_i > \sigma_j$ for all $j < i$ (resp., $j > i$). We first recall a correspondence between $S_n(123,132)$ and $C_n$ as implicitly shown in [10].

Lemma 4. ([10, Theorem 3]) There is a bijection $\varphi_1$ between $S_n(123,132)$ and $C_n$.

Proof. Given $\sigma \in S_n(123,132)$, let $\sigma_{i_1},\sigma_{i_2},\ldots,\sigma_{i_k}$ be the $k$ right-to-left maxima with $i_1 < i_2 < \cdots < i_k$. Then $c = i_1 + (i_2 - i_1) + \cdots + (i_k - i_{k-2}) + (i_k - i_{k-1})$ is a composition of $n$ since $i_k = n$. On the converse, let $m_i = n - (c_1 + \cdots + c_{i-1})$ for any given composition $n = c_1 + c_2 + \cdots + c_k \in C_n$. Set $\tau_i = m_i - 1, m_i - 2,\ldots, m_i - c_i + 1, m_i$ for $1 \leq i \leq k$. Then $c = 1 \tau_1 \tau_2 \cdots \tau_k \in S_n(123,132)$.

For example, $\sigma = 897543612$ corresponds to the composition $9 = 2 + 1 + 4 + 2$.

Given a pattern $q$, for simplicity, let $f_q(n) := \sum_{\sigma \in S_n(123,132)} f_q(\sigma)$ be the number of occurrences of pattern $q$ in $S_n(123,132)$, and we will use this notation in subsequent sections when the set in question is unambiguous. A factor of $\sigma$ is a subsequence consisting of contiguous letters in $\sigma$. From Lemma 4, we have

Proposition 5. For $n \geq 3$,

$$f_{213}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i - 1}{2}, \quad (1)$$

$$f_{231}(n) = \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=i+1}^{k-1} \sum_{j=i+1}^{k} c_j (c_i - 1). \quad (2)$$

Proof. For each permutation $\sigma \in S_n(123,132)$ with $\varphi_1(\sigma) = c_1 + c_2 + \cdots + c_k$, we can rewrite $\sigma$ as $\sigma = \tau_1 \tau_2 \cdots \tau_k$ from Lemma 4. We say that $\tau_i > \tau_j$ if all the elements in $\tau_i$ are larger than that in $\tau_j$. We see that the pattern 213 can only occur in every factor $\tau_i$ since the elements except the last one are decreasing in $\tau_i$ and $\tau_i > \tau_j$ for $j > i$. Thus, there are $\binom{c_i - 1}{2}$ choices to select two elements in $\tau_i$ to play the role of “21”, and the last element of $\tau_i$ plays the role of “3”. If $c_i \leq 2$, then there is no copy of the pattern 213 in $\tau_i$, this coincides with the value $\binom{c_i - 1}{2} = 0$ for $c_i = 1$ or 2. Summing up all the number of 213-patterns in factors $\tau_1, \tau_2,\ldots, \tau_k$ yields formula (1).

For pattern 231, we have $c_i - 1$ choices in factor $\tau_i$ to select one element to play the role of “2” and one choice (always the last element of $\tau_i$) for “3”. After this, we have $c_{i+1} + \cdots + c_k$ choices to select one element in $\tau_{i+1}, \ldots, \tau_k$ for the role of “1” since all the elements after $\tau_i$ are smaller than those in $\tau_i$. Summing up all the number of 231-patterns according to the position of “3” gives formula (2). \qed
Theorem 6. For $n \geq 3$, in the set $S_n(123,132)$, we have

\begin{align*}
    f_{213}(n) &= (n-3)2^{n-2} + 1, \\
    f_{231}(n) &= f_{312}(n) = (n^2 - 5n + 8)2^{n-3} - 1, \\
    f_{321}(n) &= (n^3/3 - 2n^2 + 14n/3 - 5)2^{n-2} + 1.
\end{align*}

\[ (3) \]

\[ (4) \]

\[ (5) \]

Proof. From $S_3(123,132) = \{213,231,312,321\}$, we have

$$f_{213}(3) = f_{231}(3) = 1.$$

To prove formula (3), Proposition 5 gives that, for $n \geq 3$,

$$f_{213}(n+1) = \sum_{c_1+c_2+\cdots+c_k=n+1}^{k} \sum_{i=1}^{k} \binom{c_i-1}{2} + \sum_{c_1+c_2+\cdots+c_k=n+1}^{k} \sum_{i=1}^{k} \binom{c_i-1}{2} = f_{213}(n).$$

If $c_k = 1$, then $k \geq 2$, and we have

$$\sum_{c_1+c_2+\cdots+c_k=n+1}^{k} \sum_{i=1}^{k} \binom{c_i-1}{2} = \sum_{c_1+c_2+\cdots+c_{k-1}=n}^{k-1} \sum_{i=1}^{k} \binom{c_i-1}{2} = f_{213}(n).$$

If $c_k \geq 2$, then we set $c_k = 1 + r_k$ with $r_k \geq 1$. From Lemma 3, we find that

$$\sum_{c_1+c_2+\cdots+c_k=n+1}^{k} \sum_{i=1}^{k} \binom{c_i-1}{2} = \sum_{c_1+c_2+\cdots+c_{k-1}+r_k=n}^{k-1} \binom{c_i-1}{2} + \binom{r_k-1}{2} + (r_k-1)$$

$$= f_{213}(n) + \sum_{c_1+\cdots+c_{k-1}+r_k=n} (r_k-1)$$

$$= f_{213}(n) + a(n) - 2^{n-1}.$$}

Combining the above two cases, we have

$$f_{213}(n+1) = 2f_{213}(n) + 2^{n-1} - 1,$$

which proves formula (3) by solving the recurrence with initial value $f_{213}(3) = 1$.

For formula (4), we first have $f_{231}(n) = f_{312}(n)$ from $231^{-1} = 312$ and $\sigma \in S_n(123,132) \iff \sigma^{-1} \in S_n(123,132)$. Using the same method as in the proof of formula (3), we can show

$$\sum_{c_1+c_2+\cdots+c_k=n+1}^{k} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j(c_i-1) = f_{231}(n) - c(n) + n2^{n-1},$$

which proves formula (4) by solving the recurrence with initial value $f_{231}(3) = 1$.
and

\[
\sum_{c_1+c_2+\cdots+c_k=n+1} c_k \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j (c_i - 1) = f_{231}(n) - a(n) - c(n) + (n + 1)2^{n-1}.
\]

It follows that, from Lemma 3,

\[
f_{231}(n + 1) = 2f_{231}(n) + (n - 2)2^{n-1} + 1.
\]

Formula (4) is proved by solving this recurrence using \(f_{231}(3) = 1\).

Since the total number of all length-3 patterns in a permutation \(\sigma \in S_n\) is \(\binom{n}{3}\), we have

\[
f_{213}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3}2^{n-1},
\]

and formula (5) holds.

The first few values of \(f_q(S_n(123, 132))\) for \(q\) of length 3 are shown below. Moreover, we observe that they appear in the On-Line Encyclopedia of Integer Sequences [15] as follows: \((f_{213}(n))_{n \geq 3}\) form sequence A000337, \((f_{231}(n))_{n \geq 3}\) form sequence A055580.

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2.2 Pattern popularity in \((132, 213)\)-avoiding permutations

We first recall a correspondence between \(S_n(132, 213)\) and \(C_n\) as follows:

**Lemma 7.** ([10, Theorem 8]) *There is a bijection \(\varphi_2\) between \(S_n(132, 213)\) and \(C_n\).*

**Proof.** Given \(\sigma \in S_n(132, 213)\), let \(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\) be the \(k\) right-to-left maxima with \(i_1 < i_2 < \cdots < i_k\). It follows that \(c = i_1 + (i_2 - i_1) + \cdots + (i_{k-1} - i_{k-2}) + (i_k - i_{k-1})\) is a composition of \(n\) since \(i_k = n\). On the converse, given a composition \(n = c_1 + c_2 + \cdots + c_k \in C_n\), let \(m_i = n - (c_1 + \cdots + c_{i-1})\) and \(\tau_i = m_i - c_i + 1, m_i - c_i + 2, \ldots, m_i - 1, m_i\) for \(1 \leq i \leq k\). Set \(\sigma = \tau_1 \tau_2 \cdots \tau_k\), and it is easy to check that \(\sigma \in S_n(132, 213)\).

For example, for the composition \(9 = 3 + 3 + 1 + 2\), we get \(\sigma = 789456312\). From this lemma, we have
Proposition 8. For \( n \geq 3 \),

\[
\begin{align*}
  f_{123}(n) &= \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k} \binom{c_i}{3}, \\
  f_{231}(n) &= \sum_{c_1+c_2+\cdots+c_k=n} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}.
\end{align*}
\]  

(6) \hspace{1cm} (7)

Proof. For a permutation \( \sigma \in S_n(132, 213) \) with \( \varphi_2(\sigma) = c_1 + c_2 + \cdots + c_k \), we rewrite \( \sigma \) as \( \sigma = \tau_1 \tau_2 \cdots \tau_k \). The pattern 123 can only occur in every factor \( \tau_i \) as \( \tau_i > \tau_j \) for \( j > i \) and the elements in \( \tau_i \) are increasing. Thus, we have \( \binom{c_i}{3} \) choices to select three elements in \( \tau_i \) to play the role of “123”, and formula (6) follows by summing up all 123-patterns in factors \( \tau_1, \tau_2, \ldots, \tau_k \).

For the pattern 231, we have \( \binom{c_i}{2} \) choices in factor \( \tau_i \) to select two elements to play the role of “23”. After this, we have \( c_i+1 + \cdots + c_k \) choices to select one element in \( \tau_{i+1}, \ldots, \tau_k \) for the role of “1” since \( \tau_j < \tau_i \) for all \( j > i \). Summing up all the number of 231-patterns according to the position of “23” gives formula (7). \( \square \)

Theorem 9. For \( n \geq 3 \), in the set \( S_n(132, 213) \), we have

\[
\begin{align*}
  f_{123}(n) &= (n - 4)2^{n-1} + n + 2, \\
  f_{231}(n) &= f_{312}(n) = (n^2 - 7n + 16)2^{n-2} - n - 4, \\
  f_{321}(n) &= (n^3/3 - 3n^2 + 38n/3 - 24)2^{n-2} + n + 6.
\end{align*}
\]  

(8) \hspace{1cm} (9) \hspace{1cm} (10)

Proof. From Proposition 8, it follows that

\[
f_{123}(n+1) = \sum_{c_1+c_2+\cdots+c_k=n+1} \sum_{i=1}^{k} \binom{c_i}{3} + \sum_{c_1+c_2+\cdots+c_k=n+1} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}.
\]

An argument similar to the proof of Theorem 6 shows that

\[
f_{123}(n+1) = 2f_{123}(n) + 2^n - n - 1.
\]

Solving this recurrence with initial value \( f_{123}(3) = 1 \) leads to formula (8).

From Lemma 1, we see that \( \sigma \in S_n(132, 213) \Leftrightarrow \sigma^{-1} \in S_n(132, 213) \), which implies \( f_{231}(n) = f_{312}(n) \) as \( 231^{-1} = 312 \).

To calculate \( f_{231}(n) \), by Proposition 8, we arrive at

\[
f_{231}(n+1) = \sum_{c_1+c_2+\cdots+c_k=n+1} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2} + \sum_{c_1+c_2+\cdots+c_k=n+1} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_j \binom{c_i}{2}.
\]
If \( c_k = 1 \), then \( k \geq 2 \), and we have

\[
\sum_{c_k=1}^{k-1} \sum_{i=1}^{k} \sum_{j=i+1}^{k} c_j \left( \frac{c_i}{2} \right) = f_{231}(n) + \alpha(n),
\]

where

\[
\alpha(n) = \sum_{c_1 + \cdots + c_k = n}^{k} \sum_{i=1}^{k} \left( \frac{c_i}{2} \right) = \sum_{c_1 + \cdots + c_k = n}^{k} \left[ \left( \frac{c_i - 1}{2} \right) + c_i - 1 \right]
\]

\[
= f_{213}(S_n(123, 132)) + \sum_{c_1 + \cdots + c_k = n}^{k} (n - k)
\]

\[
= f_{213}(S_n(123, 132)) - \alpha(n) + n2^{n-1}.
\]

Here we have used the deduced expression (1).

If \( c_k \geq 2 \), then we can derive that

\[
\sum_{c_k \geq 2}^{k-1} \sum_{i=1}^{k} \sum_{j=i+1}^{k} c_j \left( \frac{c_i}{2} \right) = f_{231}(n) + \beta(n),
\]

where

\[
\beta(n) = \sum_{c_1 + \cdots + c_k = n}^{k} \sum_{i=1}^{k-1} \left( \frac{c_i}{2} \right)
\]

\[
= \sum_{c_1 + \cdots + c_k = n}^{k} \sum_{i=1}^{k} \left( \frac{c_i}{2} \right) - \sum_{c_1 + \cdots + c_k = n}^{k} \frac{c_k(c_k - 1)}{2} = \alpha(n) - \beta(n)/2.
\]

From Lemma 3, we get

\[
f_{231}(n + 1) = 2f_{231}(n) + (2n - 6)2^{n-1} + n + 3.
\]

Formula (9) holds by solving this recurrence with initial condition \( f_{213}(3) = 1 \).

Finally, formula (10) follows from \( f_{123}(n) + 2f_{231}(n) + f_{321}(n) = \binom{n}{3}2^{n-1} \).

The first few values of \( f_q(S_n(132, 213)) \) for \( q \) of length 3 are shown below. They appear in [15] as follows: \( f_{123}(n) \) \( n \geq 3 \) form sequence A045618, \( f_{231}(n) \) \( n \geq 3 \) form sequence A055581 and \( f_{321}(n) \) \( n \geq 3 \) form sequence A055586.

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9
2.3 Pattern popularity in \((132, 231)\)-avoiding permutations

For each \(\sigma \in S_n(132, 231)\), we observe that \(n\) must lie in the beginning or the end of \(\sigma\), and \(n-1\) must lie in the beginning or the end of \(\sigma\setminus\{n\}\), and so on. Here \(\sigma\setminus\{n\}\) denotes the sequence obtained from \(\sigma\) by deleting the element \(n\). In view of such special structure, we can derive the pattern popularity in \((132, 231)\)-avoiding permutations directly.

**Theorem 10.** For \(n \geq 3\), in the set \(S_n(132, 231)\), we have

\[
f_{123}(n) = f_{213}(n) = f_{312}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3}. \tag{11}\]

**Proof.** Suppose that \(q\) is a length-3 pattern in \(\{123, 213, 312, 321\}\), and \(abc\) is a copy of the pattern \(q\). Set

\[
[n]\setminus\{a, b, c\} := \{r_1 > r_2 > \cdots > r_{n-4} > r_{n-3}\}.
\]

We will construct a permutation in the set \(S_n(132, 231)\) which contains \(abc\) as a copy of the pattern \(q\). Start with the subsequence \(\sigma^0 := abc\), and for \(i\) from 1 to \(n - 3\), \(\sigma^i\) is obtained by inserting \(r_i\) into \(\sigma^{i-1}\) such that

- If there are at least two elements in \(\sigma^{i-1}\) that are smaller than \(r_i\), then choose the two elements \(A\) and \(B\) such that \(A\) is the leftmost one and \(B\) is the rightmost one. We put \(r_i\) immediately to the left of \(A\) or immediately to the right of \(B\);

- If there is only one element \(A\) in \(\sigma^{i-1}\) such that \(A < r_i\), then we put \(r_i\) immediately to the left or to the right of \(A\);

- If all the elements in \(\sigma^{i-1}\) are larger than \(r_i\), then choose \(A\) the smallest one, and put \(r_i\) immediately to the left or to the right of \(A\).

Finally, we set \(\sigma := \sigma^{n-3}\) and \(\sigma \in S_n(132, 231)\) from the above construction. It can be seen that, the number of permutations having a copy \(abc\) is \(2^{n-3}\) since each \(r_i\) has 2 choices in the inserting procedure. Moreover, there are \(\binom{3}{n}\) choices to select three elements \(a, b, c\) as an appearance of the pattern \(q\) in \(\{123, 213, 312, 321\}\). Hence we deduce

\[
f_q(n) = \binom{n}{3} 2^{n-3}. \tag{11}
\]

Here we give an illustration for constructing a permutation in \(S_6(132, 231)\) which contains \(abc = 256\) as a copy of the pattern 123. Set \(\sigma^0 := 256\), we may have \(\sigma^1 = 8256\), \(\sigma^2 = 87256\), \(\sigma^3 = 872456\), \(\sigma^4 = 8732456\), \(\sigma := \sigma^5 = 87321456\).

We can also give a combinatorial proof for Theorem 10. Since \(\sigma \in S_n(132, 231) \iff \sigma^r \in S_n(132, 231)\), it is easy to show \(f_{123}(n) = f_{321}(n)\) and \(f_{213}(n) = f_{312}(n)\) from \(123^r = 321\) and \(213^r = 312\). It remains to give a bijection for \(f_{213}(n) = f_{123}(n)\), and our construction is motivated from Bóna [3].

We first introduce some notation about trees. A binary plane tree is a rooted unlabelled tree in which each vertex has at most two children, and each child is a left child or a right child of its parent. For each \(\sigma \in S_n(132)\), we can construct a binary plane tree \(T(\sigma)\) as follows:
the root of $T(\sigma)$ corresponds to the entry $n$ of $\sigma$, the left subtree of the root corresponds to the string of entries of $\sigma$ on the left of $n$, and the right subtree of the root corresponds to the string of entries of $\sigma$ on the right of $n$. Both subtrees are constructed recursively by the same rule. For more details, see [1, 3, 13].

A left descendant (resp., right descendant) of a vertex $x$ in a binary plane tree is a vertex in the left (resp., right) subtree of $x$. Similarly, an ascendant of a vertex $x$ in a binary plane tree is a vertex whose subtree contains $x$. Given a tree $T$ and a vertex $v \in T$, let $T_v$ be the subtree of $T$ rooted at $v$. Let $R$ be an occurrence of the pattern 123 in $\sigma \in S_n(132)$, and let $R_1, R_2, R_3$ be the three vertices of $T(\sigma)$ that correspond to $R$, going left to right. Then, $R_1$ is a left descendant of $R_2$, and $R_2$ is a left descendant of $R_3$.

According to the correspondence between 132-avoiding permutations and binary plane trees, we see that for $\sigma \in S_n(132, 231)$, $T(\sigma)$ is a binary plane tree on $n$ vertices such that each vertex has at most one child from the forbiddance of the pattern 231. For simplicity, let $T_n$ be the set of such binary plane trees on $n$ vertices such that each vertex has at most one child from the forbiddance of the pattern 231. Let $Q$ be an occurrence of the pattern 213 in $\sigma \in S_n(132, 231)$, and let $Q_2, Q_1, Q_3$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. From the characterization of trees in $T_n$, $Q_2$ is a left descendant of $Q_3$, and $Q_1$ is a right descendant of $Q_2$.

Combinatorial proof for $f_{213}(n) = f_{123}(n)$. Let $A_n$ be the set of binary plane trees in $T_n$ where three vertices forming a 213-pattern are colored black. Let $B_n$ be the set of all binary plane trees in $T_n$ where three vertices forming a 123-pattern are colored black. We define a map $\rho : A_n \rightarrow B_n$ as follows.

Given a tree $T \in A_n$ with $Q_2, Q_1, Q_3$ being the three black vertices as a 213-pattern, we define $\rho(T)$ be the tree obtained from $T$ by changing the right subtree of $Q_2$ to be its left subtree. See Figure 1 for an illustration.

![Figure 1: The bijection $\rho$.](image)

In the tree $\rho(T)$, the relative positions of $Q_2$ and $Q_3$ keep the same, and $Q_1$ is a left descendant of $Q_2$. Therefore, points $Q_1Q_2Q_3$ form a 123-pattern in $\rho(T)$, and $\rho(T) \in B_n$. On the converse, it is routine to verify that changing left subtree of $Q_2$ to be its right subtree is the desired reverse map. Therefore, $\rho$ is a bijection between $A_n$ and $B_n$.

The initial values for $f_q(S_n(132, 231))$ are

$$1, 8, 40, 160, 560, 1792, \ldots,$$

11
and this is essentially the sequence \texttt{A001789} in [15].

### 2.4 Pattern popularity in \((132,312)\)-avoiding permutations

We first present a lemma as follows:

**Lemma 11.** There is a bijection \(\varphi_4\) between \(S_n(132,312)\) and \(C_n\).

**Proof.** For \(\sigma \in S_n(132,312)\), let \(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\) be the \(k\) left-to-right maxima with \(i_1 < i_2 < \cdots < i_k\). Then \(c = (i_2 - i_1) + (i_3 - i_2) + \cdots + (i_k - i_{k-1}) + (n + 1 - i_k)\) is a composition of \(n\) since \(i_1 = 1\). On the converse, let \(n = c_k + c_{k-1} + \cdots + c_2 + c_1 \in C_n\). For \(1 \leq i \leq k\), if \(c_i = 1\) then set \(\tau_i = n - i + 1\); otherwise, set \(m_i = c_1 + \cdots + c_{i-1} - i + 2\) and \(\tau_i = n - i + 1, m_i + c_i - 2, \ldots, m_i + 1, m_i\). It is easy to get \(\sigma = \tau_k \tau_{k-1} \cdots \tau_2 \tau_1 \in S_n(132, 312)\). \(\square\)

For example, if \(9 = 3 + 1 + 2 + 3\), then \(\sigma = 6 5 4 7 8 3 9 2 1\).

**Proposition 12.** For \(n \geq 3\),

\[
 f_{123}(n) = \sum_{c_1 + c_2 + \cdots + c_k = n} \sum_{i=1}^{k-2} c_i \binom{k-i}{2}.
\]  

(12)

**Proof.** Let \(\sigma = \tau_k \cdots \tau_2 \tau_1\) be a permutation in \(S_n(132,312)\) whose composition is given by \(n = c_k + c_{k-1} + \cdots + c_2 + c_1\). It is evident that, for \(i + 1 \leq j \leq k\), the first element in \(\tau_i\) is larger than all the elements in \(\tau_j\), whereas the other elements in \(\tau_i\) are smaller than that in \(\tau_j\). Furthermore, the left-to-right maxima form an increasing subsequence and the other elements form a decreasing subsequence. Thus we have \(c_i\) choices to select one element in \(\tau_i\) to play the role of “1”, and then \(\binom{i-1}{2}\) choices to select two left-to-right maxima after \(\tau_i\) to play the role of “23”. Summing up all the number of 123-patterns in factors \(\tau_k, \ldots, \tau_2, \tau_1\) yields that

\[
 f_{123}(n) = \sum_{c_k + \cdots + c_2 + c_1 = n} \sum_{i=3}^{k} c_i \binom{i-1}{2}.
\]

By setting \(i := k - i + 1\) and using the symmetry of the summands in compositions, it is equivalent to formula (12). \(\square\)

**Theorem 13.** For \(n \geq 3\), in the set \(S_n(132, 312)\), we have

\[
 f_{123}(n) = f_{321}(n) = \binom{n}{3} 2^{n-3},
\]  

(13)

\[
 f_{213}(n) = f_{231}(n) = \binom{n}{3} 2^{n-3}.
\]  

(14)
Proof. From Lemma 1, we know that $\sigma \in S_n(132,312) \Leftrightarrow \sigma^c \in S_n(132,312)$. Hence it is obvious that $f_{123}(n) = f_{321}(n)$ and $f_{213}(n) = f_{231}(n)$ as $123^c = 321$ and $213^c = 231$.

To calculate $f_{123}(n)$, by using Proposition 12 and the similar argument in the proof of Theorem 6, we have

$$f_{123}(n + 1) = 2f_{123}(n) + (n^2 - n)2^{n-3}.$$ 

Formula (13) holds by solving the recurrence with initial value $f_{123}(3) = 1$, and formula (14) is a direct computation of $2f_{123}(n) + 2f_{213}(n) = \binom{n}{3}2^{n-1}$.  

We will give a combinatorial interpretation for $f_{231}(n) = f_{123}(n)$. For each $\sigma \in S_n(132,312)$, we construct a binary plane tree $T(\sigma)$ on $n$ vertices such that each vertex with a right descendant of some vertex does not have a left descendant from the forbiddance of the pattern 312. Let $\mathcal{T}_n$ denote the set of such trees on $n$ vertices. Let $Q$ be an occurrence of the pattern 231 in $\sigma \in S_n(132,312)$, and let $Q_2, Q_3, Q_1$ be the three vertices of $T(\sigma)$ that correspond to $Q$, going left to right. Then, $Q_2$ is a left descendant of $Q_3$, and there exists a lowest ascendant $x$ of $Q_3$ or $x = Q_3$ so that $Q_1$ is a right descendant of $x$.

**Combinatorial proof for $f_{231}(n) = f_{123}(n)$.** Let $\mathcal{A}_n$ be the set of binary plane trees in $\mathcal{T}_n$ in which three vertices forming a 231-pattern are colored black. Let $\mathcal{B}_n$ be the set of all binary plane trees in $\mathcal{T}_n$ in which three vertices forming a 123-pattern are colored black. We define a map $\varrho : \mathcal{A}_n \to \mathcal{B}_n$ as follows.

Given a tree $T \in \mathcal{A}_n$ with $Q_2, Q_3, Q_1$ being the three black vertices forming a 231-pattern, let $y$ be the parent of $x$ if it exists. We can see that $x$ is the left child of $y$ from $T \in \mathcal{A}_n$. Let $T_u := T - T_x$ be the tree obtained from $T$ by deleting the subtree $T_x$, and $T_d := T_x - T_{Q_1}$ be the tree obtained from $T_x$ by deleting $T_{Q_1}$. Now we define $\varrho(T)$ to be the tree obtained from $T$ by first adjoining $T_{Q_1}$ to the vertex $y$ as its left subtree, then adjoining $T_d$ to $Q_1$ as its left subtree and keeping all three black vertices the same if $y$ exits; otherwise, we adjoin $T_d$ to $Q_1$ as its left subtree directly. An illustration is given in Figure 2.

![Figure 2: The bijection $\varrho$.](image)

In the tree $\varrho(T)$, the relative positions of $Q_2$ and $Q_3$ are unchanged, and $Q_3$ is a left descendant of $Q_1$, thus the three black points $Q_2Q_3Q_1$ form a 123-pattern in $\varrho(T)$, and $\varrho(T) \in \mathcal{B}_n$. It is easy to describe the inverse map and we omit here.
2.5 Pattern popularity in \((132, 321)\)-avoiding permutations

We first introduce a lemma as follows:

**Lemma 14.** [14, Proposition 13] There is a bijection \(\varphi_5\) between \(S_n(132, 321)\backslash\{\text{id}\}\) and the set of 2-element subsets of [\(n\)].

**Proof.** For a permutation \(\sigma \in S_n(132, 321)\backslash\{\text{id}\}\), suppose \(\sigma_k = m\) \((k < m)\) and define \(\varphi_5(\sigma) = \{k, m\}\). On the converse, given two elements \(1 \leq k < m \leq n\), set \(\tau_1 = m - k + 1, m - k + 2, \ldots, m - 1, m, \tau_2 = 1, 2, \ldots, m - k\) and \(\tau_3 = m + 1, m + 2, \ldots, n - 1, n\). We have \(\sigma = \varphi_5^{-1}(k, m) = \tau_1 \tau_2 \tau_3\).

For example, if \(k = 4, m = 6\), then \(\sigma = 34561278\).

**Proposition 15.** For \(n \geq 3\),

\[
\begin{align*}
\mathbf{f}_{213}(n) &= \sum_{1 \leq k < m \leq n} k(m - k)(n - m), \quad (15) \\
\mathbf{f}_{312}(n) &= \sum_{1 \leq k < m \leq n} k\binom{m - k}{2}. \quad (16)
\end{align*}
\]

**Proof.** Given a permutation \(\sigma = \tau_1 \tau_2 \tau_3\) in \(S_n(132, 321)\) with \(\varphi_5(\sigma) = \{k, m\}\), we see that the elements in each \(\tau_i\) \((1 \leq i \leq 3)\) are increasing, and \(\tau_2 < \tau_1 < \tau_3\). Hence we have \(k\) choices to select one element in \(\tau_1\) to play the role of “2”, \(m - k\) choices to select one element in \(\tau_2\) to play the role of “1”, and \(n - m\) choices to select one element in \(\tau_3\) to play the role of “3”. Summing up all possible \(k\) and \(m\) gives formula (15).

For the pattern 312, we have \(k\) choices to select one element in factor \(\tau_1\) to play the role of “3”, and then have \(\binom{m - k}{2}\) choices to select two elements in factor \(\tau_2\) to play the role of “12”. Summing up all \(k\) and \(m\) proves formula (16).

We now derive the exact formulae for the popularity of patterns in \(S_n(132, 321)\) as follows.

**Theorem 16.** For \(n \geq 3\), in the set \(S_n(132, 321)\), we have

\[
\begin{align*}
\mathbf{f}_{213}(n) &= \mathbf{f}_{231}(n) = \mathbf{f}_{312}(n) = \binom{n + 2}{5}, \quad (17) \\
\mathbf{f}_{123}(n) &= n(7n^4 - 40n^3 + 85n^2 - 80n + 28)/120. \quad (18)
\end{align*}
\]

**Proof.** It is simple to prove \(\mathbf{f}_{312}(n) = \mathbf{f}_{231}(n)\) from Lemma 1 and \(312^{-1} = 231\). By Proposi-
By Proposition 15, we have
\[
f_{312}(n) = \sum_{1 \leq k < m \leq n} k \binom{m-k}{2} = \sum_{k=1}^{n-1} k \sum_{m=k+1}^{n} \binom{m-k}{2} = \sum_{k=1}^{n-1} \binom{n-k+1}{3}
\]
\[
= \sum_{k=1}^{n-1} \left[(n^3 - n)k + (1 - 3n^2)k^2 + 2nk^3 - k^4\right],
\]
and
\[
f_{213}(n) = \sum_{1 \leq k < m \leq n} k(m-k)(n-m) = \sum_{k=1}^{n-1} \sum_{m=k+1}^{n} k(m-k)(n-m)
\]
\[
= \sum_{k=1}^{n-1} \sum_{m'=1}^{n-k} km'(n-m'-k) = \sum_{k=1}^{n-1} k(n-k) \sum_{m'=1}^{n-k} m' - \sum_{k=1}^{n-1} \sum_{m'=1}^{n-k} m'^2
\]
\[
= \sum_{k=1}^{n-1} \left[(n^3 - n)k + \left(\frac{1}{6} - \frac{n^2}{2}\right)k^2 + \frac{n}{2} k^3 - \frac{1}{6} k^4\right].
\]
We get formula (17) by substituting the closed forms of \(\sum_{k=1}^{n} k^p\) (\(p = 1, 2, 3, 4\)) into the above expressions, and this theorem holds from \(2f_{231}(n) + f_{213}(n) + f_{123}(n) = \binom{n}{3} \left[\binom{n}{2} + 1\right]\).

Notice that \(f_{213}(n) = f_{231}(n)\) can be proved by Bóna’s bijection [3] on the set of binary plane trees on \(n\) vertices such that the vertex which is a right descendant of some node has no right descendant.

The first few values of \(f_q(S_n(132, 321))\) for \(q\) of length 3 are shown below, and \((f_{213}(n))_{n \geq 3}\) form sequence A000389 in [15].

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## 3 Triply restricted permutations

This section studies the pattern popularity in the permutations which avoid simultaneously any three patterns of length 3. We begin with the following proposition from [14].

**Proposition 17.** ([14, Lemma 6]) The numbers of triply restricted permutations in \(S_n\) satisfy the following equalities:
1. \(|S_n(123, 132, 213)| = |S_n(231, 312, 321)| = F_{n+1};
2. \(|S_n(123, 132, 231)| = |S_n(123, 213, 312)| = |S_n(132, 231, 312)| = |S_n(213, 312, 321)| = n;
3. \(|S_n(132, 213, 231)| = |S_n(132, 213, 312)| = |S_n(132, 231, 312)| = |S_n(213, 312, 321)| = n;
4. \(|S_n(123, 132, 312)| = |S_n(123, 213, 312)| = |S_n(132, 231, 312)| = |S_n(213, 312, 321)| = n;
5. \(|S_n(123, 231, 312)| = |S_n(132, 213, 312)| = n;
6. \(|S_n(R)| = 0 \text{ for all } R \supset \{123, 321\} \text{ if } n \geq 5, \text{ where } F_n \text{ is the Fibonacci number given by } F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.

An argument similar to the one used for doubly restricted permutations shows that we only need to consider the pattern popularity for the first set of class 1 to class 5.

### 3.1 Pattern popularity in (123, 132, 213)-avoiding permutations

It is well-known that Fibonacci number \(F_{n+1}\) counts the number of 0-1 sequences of length \(n - 1\) in which there are no consecutive ones, see [5]. We call such a sequence a Fibonacci binary word for convenience. Let \(B_n\) denote the set of all Fibonacci binary words of length \(n\). Simion and Schmidt [14] showed that

**Lemma 18.** ([14, Proposition 15*]) There is a bijection \(\psi_1\) between \(S_n(123, 132, 213)\) and \(B_{n-1}\).

**Proof.** For \(w = w_1w_2 \cdots w_{n-1} \in B_{n-1}\), we construct the permutation \(\sigma\) as follows. For \(1 \leq i \leq n - 1\), let \(X_i = [n] - \{\sigma_1, \ldots, \sigma_{i-1}\}\), and set

\[
\sigma_i = \begin{cases} 
\text{largest element in } X_i, & \text{if } w_i = 0, \\
\text{second largest element in } X_i, & \text{if } w_i = 1.
\end{cases}
\]

Finally, \(\sigma_n\) is the unique element in \(X_n\). \(\square\)

For example, if \(w = 01001010\), then \(\psi_1(w) = 9\,7\,8\,6\,4\,5\,2\,3\,1\).

Given a word \(w = w_1w_2 \cdots w_n \in B_n\), the index \(i\) (\(1 \leq i < n\)) is an ascent of \(w\) if \(w_i < w_{i+1}\). Let \(\text{asc}(w) = \{i \mid w_i < w_{i+1}\}\) be the set of ascents of \(w\), and let \(\text{maj}(w) = \sum_{i \in \text{asc}(w)} i\).

**Proposition 19.** For \(n \geq 3\),

\[
f_{312}(n) = \sum_{w \in B_{n-1}} \text{maj}(w).
\]

**Proof.** Suppose \(\sigma \in S_n(123, 132, 213)\) and \(\psi_1(\sigma) = w_1w_2 \cdots w_{n-1}\). If \(k\) is an ascent of \(w\), then \(w_kw_{k+1} = 01\) and \(\sigma_k > \sigma_{k+1}\). From bijection \(\psi_1\), we see that for all \(i \in [n-1]\), there is at most one \(j > i\) such that \(\sigma_j > \sigma_i\). This implies that \(\sigma_i > \sigma_{k+1}\) for all \(i < k\). Since \(\sigma_k\) is the largest element in \(X_k\), we have \(\sigma_i > \sigma_j\) for all \(i < k + 1\) and \(j > k + 1\). On the other hand,
since $\sigma_{k+1}$ is the second largest element in $X_{k+1}$, there exists a unique $l > k+1$ such that $\sigma_l > \sigma_{k+1}$. Thus, we find that $\sigma_l \sigma_{k+1} \sigma_l$ forms a 312-pattern for all $i \leq k$, that is the ascent $k$ will produce $k$’s copies of 312-pattern in which $\sigma_{k+1}$ plays the role of “1”. Summing up all the ascents, we derive that the number of copies of 312-pattern in $\sigma$ is $\text{maj}(\psi_1(\sigma))$. □

Recall that the generating function of the Fibonacci number $F_n$ is given by

$$\sum_{n \geq 0} F_n x^n = \frac{x}{1-x-x^2}.$$ 

Hence we can deduce that

$$\sum_{n \geq 3} F_{n+1} x^n = x \sum_{n \geq 2} F_{n+2} x^n = \frac{1}{x} \left( \frac{x}{1-x-x^2} - x - x^2 - 2x^3 \right) = \frac{x^3 (3 + 2x)}{1-x-x^2}, \quad (21)$$

$$\sum_{n \geq 2} n F_{n+2} x^n = x \left( \frac{x^2 (3 + 2x)}{1-x-x^2} \right)' = \frac{x^2 (6 + 3x - 4x^2 - 2x^3)}{(1-x-x^2)^2}, \quad (22)$$

$$\sum_{n \geq 3} \binom{n}{3} F_{n+1} x^n = \frac{x^3}{6} \left( \sum_{n \geq 3} F_{n+1} x^n \right)''' = \frac{x^3 (3 + 8x + 6x^2 + 4x^3)}{(1-x-x^2)^4}. \quad (23)$$

**Theorem 20.** For $n \geq 3$, in the set $S_n(123, 132, 213)$, we have

$$\sum_{n \geq 3} f_{231}(n) x^n = \sum_{n \geq 3} f_{312}(n) x^n = \frac{x^3 (1 + 2x)}{(1-x-x^2)^3}, \quad (24)$$

$$\sum_{n \geq 3} f_{321}(n) x^n = \frac{x^3 (1 + 6x + 12x^2 + 8x^3)}{(1-x-x^2)^4}. \quad (25)$$

**Proof.** From Lemma 1, we have $f_{231}(n) = f_{312}(n)$ as $\sigma \in S_n(123, 132, 213) \Leftrightarrow \sigma^{-1} \in S_n(123, 132, 213)$ and $231^{-1} = 312$. By Proposition 19, we can write

$$\sum_{n \geq 3} f_{312}(n) x^n = \sum_{n \geq 3} x^n \sum_{w \in B_{n-1}} \text{maj}(w) = x \sum_{n \geq 3} \sum_{w \in B_{n-1}} \text{maj}(w) x^{n-1} = xu(x),$$

where $u(x) = \sum_{n \geq 2} \sum_{w \in B_n} \text{maj}(w) x^n$. To calculate $u(x)$, we set

$$M_n(q) = \sum_{w \in B_n} q^\text{maj}(w) \text{ and } M(x, q) = \sum_{n \geq 2} M_n(q) x^n.$$ 

It is easy to get

$$u(x) = \frac{\partial M(x, q)}{\partial q} \bigg|_{q=1}.$$ 

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Given a word \( w = w_1w_2 \cdots w_n \in B_n \), if \( w_n = 0 \), then \( \text{maj}(w) = \text{maj}(w_1w_2 \cdots w_{n-1}) \); otherwise, \( w_{n-1}w_n = 01 \) and \( \text{maj}(w) = \text{maj}(w_1w_2 \cdots w_{n-2}) + n - 1 \). Hence, we have

\[
M_n(q) = M_{n-1}(q) + q^{n-1}M_{n-2}(q) \quad \text{for} \quad n \geq 4,
\]

with \( M_2(q) = 2 + q \) and \( M_3(q) = 2 + q + 2q^2 \). Multiplying the recursion by \( x^n \) and summing over \( n \geq 4 \) yields that

\[
M(x, q) - (2 + q)x^2 - (2 + q + 2q^2)x^3 = x \left[ M(x, q) - (2 + q)x^2 \right] + qx^2M(x, q).
\]

Therefore

\[
(1 - x)M(x, q) = qx^2M(x, q) + (2 + q)x^2 + 2q^2x^3.
\]

Differentiate both sides with respect to \( q \), we get

\[
(1 - x) \frac{\partial M(x, q)}{\partial q} = x^2 \left[ M(x, q) + q \frac{\partial M(x, q)}{\partial q} \right] + x^2 + 4qx^3.
\]

Setting \( q = 1 \) gives

\[
(1 - x)u(x) = x^2 \left[ M(x, 1) + \frac{\partial M(xq, q)}{\partial q} \bigg|_{q=1} \right] + x^2 + 4x^3.
\]

Notice that

\[
M(x, 1) = \sum_{n \geq 2} |B_n|x^n = \sum_{n \geq 2} F_{n+2}x^n,
\]

and

\[
\frac{\partial M(xq, q)}{\partial q} \bigg|_{q=1} = \left( \sum_{n \geq 2} \sum_{w \in B_n} (n + \text{maj}(w))q^{n+\text{maj}(w)-1}x^n \right) \bigg|_{q=1}
\]

\[
= \sum_{n \geq 2} x^n \sum_{w \in B_n} (n + \text{maj}(w))
\]

\[
= \sum_{n \geq 2} nF_{n+2}x^n + u(x).
\]

Invoking formulae (21) and (22), this implies that

\[
(1 - x)u(x) = x^2 \left[ \frac{x^2(3 + 2x)}{1 - x - x^2} + \frac{x^2(6 + 3x - 4x^2 - 2x^3)}{(1 - x - x^2)^2} + u(x) \right] + x^2 + 4x^3.
\]

Therefore, \( u(x) = x^2(1 + 2x)/(1 - x - x^3)^3 \). Multiplying \( u(x) \) by \( x \), we arrive at formula (24).

As for formula (25), we notice that

\[
\sum_{n \geq 3} f_{321}(n)x^n = \sum_{n \geq 3} \binom{n}{3} F_{n+1}x^n - 2 \sum_{n \geq 3} f_{312}(n)x^n
\]

(26)

from the observation \( 2f_{312}(n) + f_{321}(n) = \binom{n}{3} F_{n+1} \). Thus formula (25) is obtained by substituting equation (23) and the generating function of \( f_{312}(n) \) into formula (26). \( \square \)
The first few values of \( f_q(S_n(123, 132, 213)) \) for \( q \) of length 3 are shown below, and \((f_{231}(n))_{n \geq 3}\) form sequence A152881 in [15].

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3.2 Pattern popularity in other triply restricted permutations

This subsection deals with the popularity of length-3 patterns in the other four classes of triply restricted permutations. We begin with a helpful Lemma from [14] as follows:

**Lemma 21.** ([14, Proposition 16*]) We have
\[
\sigma \in S_n(123, 132, 231) \iff \sigma = n, n - 1, \ldots, k + 1, k - 1, k - 2, \ldots, 2, 1, k \text{ for some } k. \quad (27)
\]
\[
\sigma \in S_n(132, 213, 231) \iff \sigma = n, n - 1, \ldots, k + 1, 1, 2, 3, \ldots, k - 1, k \text{ for some } k. \quad (28)
\]
\[
\sigma \in S_n(123, 132, 312) \iff \sigma = n - 1, n - 2, \ldots, k + 1, n, k, k - 1, \ldots, 1 \text{ for some } k. \quad (29)
\]
\[
\sigma \in S_n(123, 231, 312) \iff \sigma = k - 1, k - 2, \ldots, 3, 2, 1, n, n - 1 \ldots, k \text{ for some } k. \quad (30)
\]

Appealing to the above structural characterizations, we can derive the pattern popularity in those classes as follows.

**Theorem 22.** For \( n \geq 3 \), in the set \( S_n(123, 132, 231) \), we have
\[
f_{213}(n) = f_{312}(n) = \binom{n}{3},
\]
\[
f_{321}(n) = (n - 2) \binom{n}{3}.
\]

**Proof.** According to the structural formula (27), the identity \( f_{213}(n) = f_{312}(n) \) can be proved by a direct bijection.

Let \( q = abc \) (\( b < a < c \)) be a copy of 213-pattern in \( \sigma \in S_n(123, 132, 231) \). We have \( \sigma(n) = c \) since \( b < c \) and \( \sigma \in S_n(123, 132, 231) \) has only one ascent at position \( n - 1 \). Therefore, \( q \) is a 213-pattern in the sole permutation
\[
\sigma = n, n - 1, \ldots, c + 1, c - 1, \ldots, a, \ldots, b, \ldots, 2, 1, c.
\]

For the sake of clarity, we underline the occurrence of the assumed pattern.

For \( q' = cba \) (312-pattern), we find similarly that \( q' \) is a 312-pattern in
\[
\sigma' = n, n - 1, \ldots, c, \ldots, a + 1, a - 1, \ldots, b, \ldots, 2, 1, a.
\]
For example, if \( n = 7 \) and \( q = 326 \), then \( \sigma = 7542316 \), \( q' = 623 \) and \( \sigma' = 7654213 \).

Hence, for every copy of 213-pattern \((q, \sigma)\), there is a unique copy of 312-pattern \((q', \sigma')\), and the converse is also true. This implies that \( f_{213}(n) = f_{312}(n) \).

To calculate \( f_{312}(n) \), we suppose \( \sigma = n, n-1, \ldots, k+1, k-1, k-2, \ldots, 2, 1, k \) for some \( k \). We construct a 312-pattern as follows: Choose one element from the first \( n-k \) elements to play the role of “3”, then choose one element from the next \( k-1 \) elements to play the role of “1”, and the last element plays the role of “2”. Thus, summing up \( k \) gives

\[
f_{312}(n) = \sum_{k=1}^{n} (n-k)(k-1) = -n^2 + (n+1) \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = \frac{n(n-1)(n-2)}{6} = \binom{n}{3}.
\]

The proof is completed by the relation \( f_{213}(n) + f_{312}(n) + f_{321}(n) = n \binom{n}{3} \).

The first few values of \( f_q(S_n(123, 132, 231)) \) for \( q \) of length 3 are shown below, and \((f_{213}(n))_{n \geq 3}\) form sequence \( A000292 \), \((f_{312}(n))_{n \geq 3}\) form sequence \( A002417 \) in [15].

<table>
<thead>
<tr>
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<th>( f_{123} )</th>
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**Theorem 23.** For \( n \geq 3 \), in the set \( S_n(123, 213, 231) \), we have

\[
f_{123}(n) = f_{312}(n) = \binom{n+1}{4},
\]

\[
f_{321}(n) = \frac{n(n-2)(n-1)^2}{12}.
\]

**Proof.** Based on structural formula (28), we could also prove \( f_{123}(n) = f_{312}(n) \) directly. Let \( abc \) be a 123-pattern in

\[
\sigma = n, n-1, \ldots, k+1, 1, \ldots, a, a+1, \ldots, b, b+1, \ldots, c-1, c, c+1, \ldots, k-1, k.
\]

Set

\[
\sigma' = n, n-1, \ldots, n-k+c, \ldots, c, 1, 2, \ldots, a, a+1, \ldots, b, b+1, \ldots, c-1.
\]

It is easy to check that \((n-k+c)a b\) is a 312-pattern of \( \sigma' \). For example, if \( \sigma = 987123456 \), then \( \sigma' = 987651234 \).

To calculate \( f_{123}(n) \), we suppose \( \sigma = n, n-1, \ldots, k+1, 1, 2, \ldots, k-1, k \) for some \( k \). A 123-pattern can be obtained by picking three elements from the last \( k \) elements to play the role of “123”. Thus, summing up all possible \( k \) gives

\[
f_{123}(n) = \sum_{k=1}^{n} \binom{k}{3} = \binom{n+1}{4}.
\]

We complete the proof from \( f_{123}(n) + f_{312}(n) + f_{321}(n) = n \binom{n}{3} \).
The first few values of \( f_q(S_n(132,213,231)) \) for \( q \) of length 3 are shown below, and \( (f_{123}(n))_{n \geq 3} \) form sequence A000332, \( (f_{321}(n))_{n \geq 3} \) form sequence A002415 in [15].

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</table>

**Theorem 24.** For \( n \geq 3 \), in the set \( S_n(123,132,312) \), we have

\[
f_{213}(n) = f_{231}(n) = \binom{n}{3}, \tag{35}
\]

\[
f_{321}(n) = (n - 2) \binom{n}{3}. \tag{36}
\]

**Proof.** In view of the structural formula (29), the equality \( f_{213}(n) = f_{231}(n) \) can be proved by a direct correspondence. Let \( abn \) be a copy of 213-pattern in

\[
\sigma = n - 1, \ldots, a, a + 1, \ldots, b, b + 1, \ldots, k + 1, n, k, k - 1, \ldots, 2, 1.
\]

Set

\[
\sigma' = n - 1, \ldots, n - a + b, \ldots, n - a + k + 1, n, n - a + k, n - a + k - 1, \ldots, n - a, \ldots, 2, 1.
\]

Then \( n - a + b, n, n - a \) is a 231-pattern of \( \sigma' \). For example, if \( \sigma = 8 7 6 5 4 9 3 2 1 \), then \( \sigma' = 8 7 6 5 4 3 7 2 1 \).

To calculate \( f_{213}(n) \), we suppose that \( \sigma = n - 1, n - 2, \ldots, k + 1, n, k, k - 1, \ldots, 2, 1 \) for some \( k \). A 213-pattern can be obtained by choosing two elements from the first \( n - k - 1 \) elements to play the role of “21”, and let \( n \) play the role of “3”. Thus, summing up all possible \( k \), we have

\[
f_{213}(n) = \sum_{k=0}^{n-1} \binom{n - k - 1}{2} = \binom{n}{3}.
\]

The proof is completed by using the relation \( f_{213}(n) + f_{231}(n) + f_{321}(n) = n \binom{n}{3} \). \( \square \)

**Theorem 25.** For \( n \geq 3 \), in the set \( S_n(123,231,312) \), we have

\[
f_{213}(n) = f_{231}(n) = \binom{n + 1}{4}, \tag{37}
\]

\[
f_{321}(n) = \frac{n(n - 2)(n - 1)^2}{12}. \tag{38}
\]

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Proof. From Lemma 1, we see that
\[ \sigma \in S_n(123, 231, 312) \iff \sigma^r \in S_n(321, 132, 213) \iff (\sigma^r)^c \in S_n(123, 231, 312). \]

As a consequence, we have \( f_{213}(n) = f_{132}(n) \) from \((213^r)^c = 312^c = 132\).

For \( f_{213}(n) \), we will employ the structure in formula (30). Suppose \( \sigma = k - 1, k - 2, \ldots, 3, 2, 1, n, n - 1 \ldots, k \) for some \( k \). A 213-pattern can be obtained as follows: Choose two elements from the first \( k - 1 \) elements to play the role of “21”, and choose one element from the last \( n - k + 1 \) elements to play the role of “3”. Thus, summing up all possible \( k \), we have
\[
f_{213}(n) = \sum_{k=1}^{n} \binom{k-1}{2}(n-k+1) = \sum_{k=0}^{n-1} \binom{k}{2}(n-k) = \binom{n+1}{4}.
\]

The formula for \( f_{321}(n) \) is obtained by the relation \( 2f_{213}(n) + f_{321}(n) = n \binom{n}{3} \). \( \Box \)

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References


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*Keywords*: permutation, pattern, composition, binary plane tree, Fibonacci number.

(Concerned with sequences A000292, A000332, A000337, A000389, A001789, A002415, A002417, A045618, A055580, A055581, A055586, and A152881.)

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