More Determinant Representations for Sequences

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Abstract

In this paper, we will find some new families of infinite (integer) matrices whose entries satisfy a non-homogeneous recurrence relation and such that the sequence of their leading principal minors is a subsequence of the Fibonacci, Lucas, Jacobsthal, or Pell sequences.
1 Introduction

Throughout this paper, unless noted otherwise, we will use the following notation. Let $\alpha = (\alpha_i)_{i \geq 0}$ and $\beta = (\beta_i)_{i \geq 0}$ be two arbitrary sequences starting with a common first term $\alpha_0 = \beta_0$. We denote by $P_{\alpha,\beta}(n)$ the generalized Pascal triangle associated with the sequences $\alpha$ and $\beta$, which is introduced as follows. Actually, $P_{\alpha,\beta}(n) = [P_{i,j}]_{0 \leq i,j \leq n}$ is a square matrix of order $n + 1$ whose $(i,j)$-entry $P_{i,j}$ obeys the following rules:

$$P_{i,0} = \alpha_i, \quad P_{0,j} = \beta_j \quad \text{for} \; i,j = 0,1,2,\ldots,n, \quad \text{and} \quad P_{i,j} = P_{i,j-1} + P_{i-1,j} \quad \text{for} \; 1 \leq i,j \leq n.$$

We also denote by $T_{\alpha,\beta}(n) = [T_{i,j}]_{0 \leq i,j \leq n}$ the Toeplitz matrix of order $n + 1$ whose $(i,j)$-entry $T_{i,j}$ obeys the following rules:

$$T_{i,0} = \alpha_i, \quad T_{0,j} = \beta_j \quad \text{for} \; i,j = 0,1,2,\ldots,n, \quad \text{and} \quad T_{i,j} = T_{k,l} \quad \text{if} \; i-j = k-l.$$

The unipotent lower triangular matrix $L(n) = [L_{i,j}]_{0 \leq i,j \leq n}$ is again a square matrix of order $n + 1$ with entries:

$$L_{i,j} = \begin{cases} 0, & \text{if} \; 0 \leq i < j \leq n; \\ \binom{i}{j}, & \text{if} \; 0 \leq j \leq i \leq n. \end{cases}$$

We put $U(n) = L(n)^t$, where $A^t$ signifies the transpose of matrix $A$. Moreover, a lower Hessenberg matrix $H(n) = [H_{i,j}]_{0 \leq i,j \leq n}$ is a square matrix of order $n + 1$, where $H_{i,j} = 0$ whenever $j > i + 1$ and $H_{i,i+1} \neq 0$ for some $i$, $0 \leq i \leq n - 1$.

Given a matrix $A$, we denote by $R_i(A)$ (resp., $C_j(A)$) the row $i$ (resp., the column $j$) of $A$. We also denote by $A^{[1]}$ the submatrix obtained from $A$ by deleting the first column of $A$.

Given a sequence $\varphi = (\varphi_i)_{i \geq 0}$, define the binomial transform of $\varphi$ to be the sequence $\hat{\varphi} = (\hat{\varphi}_i)_{i \geq 0}$ with

$$\hat{\varphi}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \varphi_k.$$ 

The Fibonacci sequence (A000045 in [3]) is defined by the recurrence relation:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \; n \geq 2.$$ 

The Lucas sequence (A000032 in [3]) is defined by the recurrence relation:

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \; n \geq 2.$$ 

The Jacobsthal sequence (A001045 in [3]) is defined by the recurrence relation:

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for} \; n \geq 2.$$ 

The Pell sequence (A000129 in [3]) is defined by the recurrence relation:

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for} \; n \geq 2.$$
Let $A = [A_{i,j}]_{i,j \geq 0}$ be an arbitrary infinite matrix. We denote the elementary row operation of type three by $O_{r,s}(\lambda)$, where $r \neq s$ and $\lambda$ a scalar, that is

$$R_k(O_{r,s}(\lambda)A) = \begin{cases} \lambda R_s(A) & \text{if } k = r; \\ R_k(A) & \text{if } k \neq r. \end{cases}$$

The $n$th leading principal minor of $A$, denoted by $d_n(A)$, is defined as follows:

$$d_n(A) = \det[A_{i,j}]_{0 \leq i,j \leq n}, \quad (n = 0, 1, 2, 3, \ldots).$$

We put $D(A) = (d_n(A))_{n \geq 0}$. Two infinite matrices $A$ and $B$ are said to be equimodular if $D(A) = D(B)$. Given a sequence $\omega = (\omega_n)_{n \geq 0}$, a family $\{A_t\}_{t \in I}$ of equimodular matrices are said to be $\omega$-equimodular if $D(A_t) = \omega$ for all $t \in I$. We will denote the family of $\omega$-equimodular matrices by $A_\omega$. The infinite matrices in $A_\omega$ are said to be determinant representations of $\omega$. Note that for any sequence $\omega = (\omega_n)_{n \geq 0}$, there is a determinant representation of $\omega$, in other words $A_\omega \neq \emptyset$. Indeed, expanding along the last rows, it is easy to see that

$$\begin{pmatrix} \omega_0 & 1 & * & * & \cdots \\ -\omega_1 & 0 & 1 & * & \cdots \\ \omega_2 & 0 & 0 & 1 & \cdots \\ -\omega_3 & 0 & 0 & 0 & 1 \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in A_\omega,$$

(see also Theorem 3.2 and the Remark after this theorem in [4]). Especially, there are many different determinant representations of $\omega$, when $\omega$ is a (sub-)sequence of Fibonacci, Lucas, Jacobsthal and Pell sequences. Some examples of such matrices can be found in [1, 2].

In this paper, we are going to find some determinant representations of the sequences:

$$\mathcal{F} = (F_{n+1})_{n \geq 0}, \quad \mathcal{L} = (L_{n+1})_{n \geq 0}, \quad \mathcal{J} = (J_{n+1})_{n \geq 0} \quad \text{and} \quad \mathcal{P} = (P_{n+1})_{n \geq 0}.$$

It is worthwhile to point out that we will use non-homogeneous recurrence relations to construct these determinant representations.

In the sequel, we introduce a new family of (infinite) matrices $A(\infty) = [A_{i,j}]_{i,j \geq 0}$, whose entries obey a non-homogeneous recurrence relation. Actually, for two constants $u$ and $v$, and arbitrary sequences $\lambda = (\lambda_i)_{i \geq 0}$ and $\mu = (\mu_i)_{i \geq 0}$ with $\mu_0 = 0$, the first column and row of matrix $A(\infty)$ are the sequences

$$(A_{i,0})_{i \geq 0} = (\lambda_0, \lambda_1, \lambda_2, \ldots, A_{i,0} = \lambda_i, \ldots),$$

and

$$(A_{0,j})_{j \geq 0} = (\lambda_0, \lambda_0 + u, \lambda_0 + 2u, \ldots, A_{0,j} = \lambda_0 + ju, \ldots),$$

respectively, while the remaining entries $A_{i,j}$ ($i,j \geq 1$) are obtained from the following non-homogeneous recurrence relation:

$$A_{i,j} = A_{i,j-1} + A_{i-1,j} - \lambda_{i-1} + \mu_i - \mu_{i-1} + (j-1)(v-u), \quad i,j \geq 1.$$
We denote by $A(n)$ the submatrix of $A(\infty)$ consisting of the entries in its first $n + 1$ rows and columns. The matrix $A(3)$, for example, is then given by

$$A(3) = \begin{pmatrix}
\lambda_0 & \lambda_0 + u & \lambda_0 + 2u & \lambda_0 + 3u \\
\lambda_1 & \lambda_1 + \mu_1 + u & \lambda_1 + 2\mu_1 + 2u + v & \lambda_1 + 3\mu_1 + 3u + 3v \\
\lambda_2 & \lambda_2 + \mu_2 + u & \lambda_2 + 2\mu_2 + \mu_1 + 2u + 2v & \lambda_2 + 3\mu_2 + 3\mu_1 + 3u + 7v \\
\lambda_3 & \lambda_3 + \mu_3 + u & \lambda_3 + 2\mu_3 + \mu_2 + \mu_1 + 2u + 3v & \lambda_3 + 3\mu_3 + 3\mu_2 + 4\mu_1 + 3u + 12v
\end{pmatrix}.$$ 

Finally, the main result of this paper can be stated as follows:

**Main Theorem.** The matrix $A(n)$, $n \geq 0$, defined as above, satisfies the following statements:

(a) $A(n) = L(n) \cdot H(n) \cdot U(n)$, where

$$H(n) = \begin{pmatrix}
\hat{\lambda}_0 & u & 0 & \cdots & 0 \\
\lambda_1 & \hat{\lambda}_1 & u & 0 & \cdots \\
\lambda_2 & \hat{\lambda}_2 & u & \lambda_1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\lambda_n & \hat{\lambda}_n & T(\hat{\mu}_1, \hat{\mu}_2, \ldots) & (\hat{\mu}_1, v, 0, 0, \ldots) & (n - 1)
\end{pmatrix}. $$

In particular, we have $\det(A(n)) = \det(H(n))$. 

(b) In the case when $u = v = 1$ and $\lambda_i = (2^i - 1)c + 1$, we have the following statements:

(b. 1) if $\mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right)c - \frac{i(i-3)}{2}$, then $\det(A(n)) = F_{n+1}$.

(b. 2) if $\mu_i = \left(\frac{5+3i}{4} - 2^i - \frac{2i+1}{4}\right)c + \frac{5(3^i-1)}{4} + \frac{i}{2}$, then $\det(A(n)) = L_{n+1}$.

(b. 3) if $\mu_i = i^2 c - i^2 + 2i$, then $\det(A(n)) = J_{n+1}$.

(b. 4) if $\mu_i = \left(\frac{2^{i+1} + \frac{(i+1)(i-4)}{2}}{2}\right)c + \frac{(5-i)i}{2}$, then $\det(A(n)) = P_{n+1}$.

As mentioned previously, we have obtained some determinant representations of the sequences:

$\mathcal{F} = (F_{n+1})_{n \geq 0}$, $\mathcal{L} = (L_{n+1})_{n \geq 0}$, $\mathcal{J} = (J_{n+1})_{n \geq 0}$ and $\mathcal{P} = (P_{n+1})_{n \geq 0}$. 

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which are presented in the following:

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots \\
c + 1 & 2c + 2 & 3c + 6 & \cdots \\
3c + 1 & 7c + 3 & 12c + 8 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in A_F, \quad \begin{pmatrix}
1 & 2 & 3 & \cdots \\
c + 1 & 2c + 5 & 3c + 10 & \cdots \\
3c + 1 & 9c + 13 & 16c + 30 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in A_L,
\]

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots \\
c + 1 & 2c + 3 & 3c + 6 & \cdots \\
3c + 1 & 7c + 2 & 12c + 6 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in A_J \quad \text{and} \quad \begin{pmatrix}
1 & 2 & 3 & \cdots \\
c + 1 & 2c + 4 & 3c + 8 & \cdots \\
3c + 1 & 8c + 5 & 14c + 3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \in A_P.
\]

\section{Main results}

As the first result of this paper, we consider the following theorem.

\textbf{Theorem 1.} For two arbitrary sequences \((\lambda_i)_{i \geq 0}\) and \((\mu_i)_{i \geq 0}\), with \(\mu_0 = 0\), and some integers \(u\) and \(v\), let \(A(\infty) = [A_{i,j}]_{i,j \geq 0}\) be an infinite dimensional matrix whose entries are given by

\[A_{i,j} = A_{i,j-1} + A_{i-1,j} - \lambda_{i-1} + \mu_i - \mu_{i-1} + (j - 1)(v - u), \quad i, j \geq 1\]

and the initial conditions \(A_{i,0} = \lambda_i\) and \(A_{0,i} = \lambda_0 + iu\), \(i \geq 0\). If \(A(n) = [A_{i,j}]_{0 \leq i,j \leq n}\), then we have

\[A(n) = L(n) \cdot H(n) \cdot U(n),\]

where

\[
H(n) = \begin{pmatrix}
\hat{\lambda}_0 & u & 0 & \cdots & 0 \\
\hat{\lambda}_1 & & & & \\
\vdots & & & & \\
\hat{\lambda}_n & & & & \\
& T_{(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots)}(\hat{\mu}_1,v,0,0,\ldots)(n - 1)
\end{pmatrix}
\]

\textbf{Proof.} First of all, we recall that the entries of \(L(n) = [L_{i,j}]_{0 \leq i,j \leq n}\) satisfy the following recurrence

\[L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 1 \leq i, j \leq n.\]

Similarly, for the entries of \(U(n) = [U_{i,j}]_{0 \leq i,j \leq n}\) we have

\[U_{i,j} = U_{i-1,j-1} + U_{i,j-1}, \quad 1 \leq i, j \leq n.\]
In what follows, for convenience, we will let $A = A(n)$, $L = L(n)$, $H = H(n)$ and $U = U(n)$. Now, for the proof of the desired factorization we compute the $(i, j)$-entry of $L \cdot H \cdot U$, that is

$$
(L \cdot H \cdot U)_{i,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,j}.
$$

(5)

In fact, we should establish

$$
R_0(L \cdot H \cdot U) = R_0(A) = (\lambda_0, \lambda_0 + u, \ldots, \lambda_0 + nu),
$$

$$
C_0(L \cdot H \cdot U) = C_0(A) = (\lambda_0, \lambda_1, \ldots, \lambda_n),
$$

and finally, show that

$$
(L \cdot H \cdot U)_{i,j} = (L \cdot H \cdot U)_{i-1,j-1} + (L \cdot H \cdot U)_{i-1,j} - \lambda_{i-1} - \mu_i - \mu_{i-1} + (j - 1)(v - u),
$$

(6)

for $1 \leq i, j \leq n$.

Let us do the required calculations. Assume first that $i = 0$. Then, we have

$$
(L \cdot H \cdot U)_{0,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{0,r} H_{r,s} U_{s,j} = \sum_{s=0}^{n} H_{0,s} U_{s,j} = H_{0,0} U_{0,j} + H_{0,1} U_{1,j} = \lambda_0 + ju,
$$

and so $R_0(L \cdot H \cdot U) = R_0(A) = (\lambda_0, \lambda_0 + u, \ldots, \lambda_0 + nu)$.

Assume next that $j = 0$. In this case, we obtain

$$
(L \cdot H \cdot U)_{i,0} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,0} = \sum_{r=0}^{n} L_{i,r} H_{r,0} = \sum_{r=0}^{n} \left( \sum_{s=0}^{i} \lambda_r \right),
$$

and hence we have $C_0(L \cdot H \cdot U) = C_0(A) = (\lambda_0, \lambda_1, \ldots, \lambda_n)$.

Finally, we must establish (6). Let us for the moment assume that $1 \leq i, j \leq n$. In this case, we have

$$
(L \cdot H \cdot U)_{i,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,j} = \sum_{r=0}^{n} L_{i,r} H_{r,0} U_{0,j} + \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j}.
$$

(7)

Let $\Omega(i, j) = \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j}$. Then, using (4), we obtain

$$
\Omega(i, j) = \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} (U_{s-1,j-1} + U_{s,j-1}) = \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s-1,j-1} + \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j-1}
$$

$$
= \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s-1,j-1} + (L \cdot H \cdot U)_{i,j-1} + \sum_{s=1}^{n} L_{i,0} H_{0,s} U_{s-1,j-1} - \sum_{r=0}^{n} L_{i,r} H_{r,0} U_{0,j-1}
$$

(8)
For convenience, we write \( \Theta(i, j) = \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s-1,j-1} \). Now, we apply (3), to get

\[
\Theta(i, j) = \sum_{r=1}^{n} \sum_{s=1}^{n} (L_{i-1,r-1} + L_{i-1,r}) H_{r,s} U_{s-1,j-1}
\]

\[
= \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r-1} H_{r,s} U_{s-1,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s-1,j-1}
\]

\[
= \sum_{r=2}^{n} \sum_{s=2}^{n} L_{i-1,r-1} H_{r,s} U_{s-1,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r-1} H_{r,1} U_{0,j-1}
\]

\[
+ \sum_{s=2}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s-1,j-1}
\]

\[
= \sum_{r=2}^{n} \sum_{s=2}^{n} L_{i-1,r-1} H_{r,s} U_{s-1,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r-1} H_{r,1} U_{0,j-1}
\]

\[
+ \sum_{s=2}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s,j}
\]

\[
- \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s,j-1} \quad \text{(by the structure of } H) \]

\[
= \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s,j-1} + \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r-1} H_{r,1} U_{0,j-1} + \sum_{s=1}^{n} \sum_{s=1}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1}
\]

\[
+ \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s,j} - \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,0} U_{0,j} - \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i-1,r} H_{r,s} U_{s,j-1}
\]

(note that \( L_{i-1,n-1} = U_{n-1,j-1} = 0 \))

\[
= \sum_{r=1}^{n} L_{i-1,r-1} H_{r,1} U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1} + \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i-1,r} H_{r,s} U_{s,j}
\]

\[
- \sum_{s=0}^{n} L_{i-1,0} H_{0,s} U_{s,j} - \sum_{r=1}^{n} L_{i-1,r} H_{r,0} U_{0,j}
\]

\[
= \sum_{r=1}^{n} L_{i-1,r-1} H_{r,1} U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1} + (L \cdot H \cdot U)_{i-1,j}
\]

\[
- \sum_{s=0}^{n} L_{i-1,0} H_{0,s} U_{s,j} - \sum_{r=1}^{n} L_{i-1,r} H_{r,0} U_{0,j} \quad \text{(by } (5)).
\]
By substituting this in (8), we obtain
\[
\Omega(i, j) = (L \cdot H \cdot U)_{i,j-1} + (L \cdot H \cdot U)_{i-1,j} - \sum_{s=0}^{n} L_{i-1,0} H_{s,sij} - \sum_{r=1}^{n} L_{i-1,r} H_{r,00j} + \sum_{s=1}^{n} L_{i,0} H_{s,0sij} - \sum_{r=0}^{n} L_{i,r} H_{r,00j-1}.
\]

Finally, if the above expression is substituted in (7) and the sums are put together, then we obtain
\[
(L \cdot H \cdot U)_{i,j} = (L \cdot H \cdot U)_{i-1,j} + (L \cdot H \cdot U)_{i,j-1} + \Psi(i, j),
\]
where
\[
\Psi(i, j) := \sum_{r=0}^{n} L_{i,r} H_{r,00j} + \sum_{r=1}^{n} L_{i-1,r-1} H_{r,00j} + \sum_{s=1}^{n} L_{i-1,0} H_{s,sij} - \sum_{r=1}^{n} L_{i-1,r} H_{r,00j} + \sum_{s=1}^{n} L_{i,0} H_{s,0sij} - \sum_{r=0}^{n} L_{i,r} H_{r,00j-1}.
\]

However, by easy calculations one can show that
\[
\sum_{r=0}^{n} L_{i,r} H_{r,00j} - \sum_{r=0}^{n} L_{i,r} H_{r,00j-1} = 0,
\]
\[
\sum_{r=1}^{n} L_{i-1,r-1} H_{r,00j} = \sum_{r=1}^{n} \binom{i-1}{r-1} \hat{\mu}_{r} = \sum_{r=1}^{n} \left( \binom{i}{r} - \binom{i-1}{r} \right) \hat{\mu}_{r} = \mu_{i} - \mu_{i-1},
\]
\[
\sum_{r=1}^{n} L_{i-1,r} H_{r,00j} = \sum_{r=0}^{n} \hat{\lambda}_{r} - \lambda_{0} = \lambda_{i-1} - \lambda_{0},
\]
\[
\sum_{s=2}^{n} L_{i-1,0} H_{1,s} U_{s-1,j-1} = (j-1)v,
\]
\[
\sum_{s=0}^{n} L_{i-1,0} H_{0,s} U_{s,j} = \lambda_{0} + ju,
\]
\[
\sum_{s=1}^{n} L_{i,0} H_{0,s} U_{s-1,j-1} = u,
\]
and so
\[
\Psi(i, j) = \mu_{i} - \mu_{i-1} - \lambda_{i-1} + (j-1)(v-u).
\]

This completes the proof.
Before stating the next result, we need to introduce some additional definitions. Let 
\( \lambda = (\lambda_i)_{i \geq 0} \) and \( \mu = (\mu_i)_{i \geq 0} \) be two arbitrary sequences. The \textit{convolution} of \( \lambda \) and \( \mu \) is the sequence \( \nu = (\nu_i)_{i \geq 0} \), where
\[
\nu_i = \sum_{k=0}^{i} \lambda_k \mu_{i-k}.
\]
The \textit{convolution matrix} associated with sequences \( \lambda \) and \( \mu \) is the infinite matrix \( A(\infty) \) whose first column \( C_0(A(\infty)) \) is \( \lambda \) and whose \( j \)th column \( (j = 1, 2, \ldots) \) is the convolution of sequences \( C_{j-1}(A(\infty)) \) and \( \mu \). We say that the convolution matrix of the sequences \( \lambda \) and \( \lambda \) is the convolution matrix of the sequence \( \lambda \).

There are many well-known integer matrices which can be written as convolution matrices of some sequences. For instance, \( U(\infty) \) is the convolution matrix of the sequences \((1, 0, 0, \ldots)\) and \((1, 1, 0, 0, \ldots)\) and \( P_{(1,1,\ldots),(1,1,\ldots)}(\infty) \) is the convolution matrix of the sequence \((1, 1, \ldots)\).

We will need the following technical result \cite[Theorem 3.1]{4}.

\textbf{Proposition 2.} Let
\[
A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n, \quad V(x) = \sum_{n=0}^{\infty} v_n x^n \quad \text{and} \quad W(x) = \sum_{n=0}^{\infty} w_n x^n
\]
be the generating functions for the sequences \((a_n)_{n \geq 1}\), \((b_n)_{n \geq 0}\), \((v_n)_{n \geq 0}\), and \((w_n)_{n \geq 0}\), respectively. Consider an infinite dimensional matrix of the following form:
\[
M(\infty) = \begin{pmatrix} b_0 & v_0 & v_0 w_0 & \cdots \\ b_1 & v_1 & v_0 w_1 + v_1 w_0 & \cdots \\ b_2 & v_2 & v_0 w_2 + v_1 w_1 + v_2 w_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]
where \( C_0(M(\infty)) = (b_0, b_1, \ldots)^t \) and \( M(\infty)^{[1]} \) is the convolution matrix of the sequences \((v_i)_{i \geq 0}\) and \((w_j)_{j \geq 0}\). If
\[
A(W(x)) = B(x)/V(x),
\]
then for any non-negative integer \( n \), there holds
\[
\det(M(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1},
\]
where \( M(n) \) is the \((n+1) \times (n+1)\) upper left corner matrix of \( M(\infty) \).

We are now in a position to prove the following theorem which is the second result of this paper.

\textbf{Theorem 3.} Let \( A(n) \) be defined as in Theorem 1 and let \( c \) be a constant. In the case when \( u = v = 1 \) and \( \lambda_i = (2^i - 1)c + 1 \), we have the following statements:

\[ \]
(a) if \( \mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right) c - \frac{i(i-3)}{2} \), then \( \det(A(n)) = F_{n+1} \).

(b) if \( \mu_i = \left(\frac{5\cdot3}{4} - 2^i - \frac{2i+1}{4}\right) c + \frac{5(3^i-1)}{4} + \frac{i}{2} \), then \( \det(A(n)) = L_{n+1} \).

(c) if \( \mu_i = i^2 c - i^2 + 2i \), then \( \det(A(n)) = J_{n+1} \).

(d) if \( \mu_i = \left(2^{i+1} + \frac{(i+1)(i-4)}{2}\right) c + \frac{(5-i)}{2} \), then \( \det(A(n)) = P_{n+1} \).

**Proof.** Let \( \mu = (\mu_i)_{i\geq 0} \) be a sequence with \( \mu_0 = 0 \) and let \( c \) be a constant. Let \( \lambda = (\lambda_i)_{i\geq 0} \) be a sequence with \( \lambda_i = (2^i - 1)c + 1 \). We consider the infinite matrices

\[
A_{\infty} = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & c & c & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c & \cdots & \cdots & \cdots & c \\
T(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots,\hat{\mu}_1,\hat{\mu}_2,0,0,\ldots)(n-1)
\end{pmatrix}
\]

with the initial conditions \( A_{i,0} = (2^i - 1)c + 1 \) and \( A_{0,i} = 1 + i \). By Theorem 2, we observe that

\[
A(n) = L(n) \cdot H(n) \cdot U(n),
\]

where

\[
H(n) = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & c & c & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c & \cdots & \cdots & \cdots & c \\
T(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots,\hat{\mu}_1,\hat{\mu}_2,0,0,\ldots)(n-1)
\end{pmatrix}
\]

Evidently \( \det(A(n)) = \det(H(n)) \), so it suffices to find \( \det(H(n)) \). From the structure of matrix \( H(\infty) \), we have

\[
C_0(H(\infty)) = (b_i)_{i\geq 0} = (1, c, c, \ldots)^t,
\]

whose generating function is

\[
B(x) = \frac{1 + (c-1)x}{1-x}.
\]

(a) Let \( \mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right) c - \frac{i(i-3)}{2} \). In this case, we have the following infinite dimensional matrices:

\[
A(\infty) = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots \\
c + 1 & 2c + 3 & 3c + 6 & 4c + 10 & \cdots \\
3c + 1 & 7c + 3 & 12c + 8 & 18c + 17 & \cdots \\
7c + 1 & 17c + 2 & 32c + 8 & 53c + 23 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
and
\[ H(\infty) = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
\frac{c}{1} & 2c + 5 & 3c + 10 & 4c + 16 & \cdots \\
\frac{c}{2} & 9c + 13 & 16c + 30 & 24c + 53 & \cdots \\
\frac{c}{3} & 31c + 36 & 62c + 88 & 101c + 163 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Note that the submatrix \( H(\infty)[1] \) is the convolution of sequences
\[ (v_i)_{i \geq 0} = (1, c + 1, 2c - 1, c, c, \ldots), \quad \text{and} \quad (w_i)_{i \geq 0} = (0, 1, 0, 0, 0, \ldots), \]
whose generating functions are
\[ V(x) = \frac{1 + cx + (c - 2)x^2 - (c - 1)x^3}{1 - x} \quad \text{and} \quad W(x) = x, \]
respectively. Plugging these generating functions into (9) yields
\[ A(W(x)) = A(x) = \frac{1 + (c-1)x}{1-x} = 1 - x + 2x^2 - 3x^3 + \cdots + (-1)^n F_{n+1} x^n + \cdots, \]
and it follows by Proposition 2 that
\[ \det(H(n)) = (-1)^n v_0^{n+1} w_1^{(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = F_{n+1}, \]
as required.

(b) Let \( \mu_i = \left( \frac{5\cdot3^i}{4} - \frac{2i}{4} - \frac{2i+1}{4} \right) c + \frac{5(3^i-1)}{4} + \frac{i}{2} \). The infinite dimensional matrices created in this case are as follows:
\[ A(\infty) = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots \\
c + 1 & 2c + 5 & 3c + 10 & 4c + 16 & \cdots \\
3c + 1 & 9c + 13 & 16c + 30 & 24c + 53 & \cdots \\
7c + 1 & 31c + 36 & 62c + 88 & 101c + 163 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \]
and
\[ H(\infty) = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
\frac{c}{1} & 2c + 5 & 3c + 10 & 4c + 16 & \cdots \\
\frac{c}{2} & 9c + 13 & 16c + 30 & 24c + 53 & \cdots \\
\frac{c}{3} & 31c + 36 & 62c + 88 & 101c + 163 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
Again, one can easily see that the submatrix \( H(\infty)[1] \) is the convolution of sequences
\[ (v_i)_{i \geq 0} = (1, c + 3, 4c + 5, 9c + 10, 19c + 20, \ldots), \]
(with general form $v_0 = 1$, $v_1 = c + 3$ and $v_i = (5 \cdot 2^{i-2})(c + 1) - c$ for $i \geq 2$), and

$$(w_i)_{i \geq 0} = (0, 1, 0, 0, \ldots).$$

The generating functions for these sequences are

$$V(x) = \frac{(1 + (c-1)x)(-x^2 + x + 1)}{(1-x)(1-2x)}, \quad \text{and} \quad W(x) = x,$$

respectively. If $B(x)$, $V(x)$ and $W(x)$ are substituted in (9), then we obtain

$$A(W(x)) = A(x) = \frac{1+(c-1)x}{1-x} = 1 - 3x + 4x^2 - 7x^3 + 11x^4 + \cdots + (-1)^n L_{n+1} x^n + \cdots,$$

and by Proposition 2, it follows that

$$\det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = L_{n+1},$$

as required.

(c) Let $\mu_i = i^2 c - i^2 + 2i$. In this case, we have the following infinite dimensional matrices:

$$A(\infty) = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots \\
c + 1 & 2c + 3 & 3c + 6 & 4c + 10 & \cdots \\
3c + 1 & 7c + 2 & 12c + 6 & 18c + 14 & \cdots \\
7c + 1 & 16c - 1 & 30c + 1 & 50c + 11 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and

$$H(\infty) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
c & T(c+1,2c-2,0,0,\ldots),(c+1,1,0,0,\ldots) (\infty) \\
\vdots & \vdots & \ddots & \vdots
\end{pmatrix}.$$ 

Moreover, from the structure of $H(\infty)$, we see that the submatrix $H(\infty)^{[1]}$ is the convolution of sequences

$$(v_i)_{i \geq 0} = (1, c+1, 2c - 2, 0, 0, \ldots), \quad \text{and} \quad (w_i)_{i \geq 0} = (0, 1, 0, 0, \ldots),$$

with generating functions $V(x) = 1 + (c+1)x + (2c-2)x^2$ and $W(x) = x$, respectively. Substituting the obtained generating functions in (9), we obtain

$$A(W(x)) = A(x) = \frac{1+(c-1)x}{1-x} = 1 - 3x + 4x^2 - 5x^3 + \cdots + (-1)^n J_{n+1} x^n + \cdots.$$
Therefore, it follows from Proposition 2 that

\[ \det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = J_{n+1}, \]

as required.

(d) Let \( \mu_i = (2^{i+1} + \frac{(i+1)(i-4)}{2}) c + \frac{(5-i)i}{2} \). This time, we will deal with the following matrices:

\[
A(\infty) = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots \\
0 & c+1 & 2c+4 & 3c+8 & 4c+13 & \cdots \\
0 & 3c+1 & 8c+5 & 14c+13 & 21c+26 & \cdots \\
0 & 7c+1 & 21c+5 & 41c+17 & 68c+42 & \cdots \\
0 & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

and

\[
H(\infty) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & c & 0 & \cdots & 0 \\
c & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}.
\]

In addition, the submatrix \( H(\infty)^{[1]} \) of \( H(\infty) \) is the convolution of sequences:

\((v_i)_{i \geq 0} = (1, c + 2, 3c - 1, 2c, 2c, \ldots)\) and \((w_i)_{i \geq 0} = (0, 1, 0, 0, \ldots)\).

Note that the generating functions of these sequences are

\[ V(x) = \frac{1 + (1 + c)x + (2c - 3)x^2 - (c - 1)x^3}{1 - x} \] and \[ W(x) = x, \]

respectively. After having substituted these generating functions in (9), we obtain

\[ A(W(x)) = A(x) = \frac{1+(c-1)x}{1-(1+c)x+(2c-3)x^2-(c-1)x^3} = 1-2x+5x^2-12x^3+\cdots+(-1)^n P_{n+1} x^{n+\cdots} \]

Now, by Proposition 2, we deduce that

\[ \det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = P_{n+1}, \]

as required.

This completes the proof. \(\Box\)
3 Some remarks

In this section, we will explain how the sequences \((\lambda_i)_{i \geq 0}\) and \((\mu_i)_{i \geq 0}\) in Theorem 3, are determined. Consider the following lower Hessenberg matrix

\[
H(\infty) = [H_{i,j}]_{i,j \geq 0} = \begin{pmatrix}
    h_{0,0} & h_{0,1} & 0 & 0 & 0 & \cdots \\
    h_{1,0} & h_{1,1} & h_{1,2} & 0 & 0 & \cdots \\
    h_{2,0} & h_{2,1} & h_{1,1} & h_{1,2} & 0 & \cdots \\
    h_{3,0} & h_{3,1} & h_{2,1} & h_{1,1} & h_{1,2} & \cdots \\
    h_{4,0} & h_{4,1} & h_{3,1} & h_{2,1} & h_{1,1} & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Let \(H(n) = [H_{i,j}]_{0 \leq i,j \leq n}\), and let \(d_n\) be the \(n\)th determinant of \(H(n)\). In what follows, we show that the sequence of principal minors of \(H(\infty)\), i.e., \(D(H(\infty)) = (d_n)_{n \geq 0}\), satisfies a recurrence relation.

**Proposition 4.** With the above notation, we have

\[
d_n = \begin{cases}
    h_{0,0}, & \text{if } n = 0, \\
    (-1)^n h_{0,1}(h_{1,2})^{n-1} h_{n,0} + \sum_{k=0}^{n-1} h_{n-k,1} (-1)^{n-k-1} d_k, & \text{if } n \geq 1.
\end{cases}
\]

**Proof.** Obviously, \(d_0 = h_{0,0}\). Hence, from now on we assume \(n > 1\). First, we apply the following row operations:

\[
H_1(n) = \left( \prod_{i=1}^{n} O_{i,0} \left( -\frac{h_{i,1}}{h_{0,1}} \right) \right) H(n),
\]

\[
H_2(n) = \left( \prod_{i=1}^{n-1} O_{i+1,1} \left( -\frac{h_{i,1}}{h_{1,2}} \right) \right) H_1(n),
\]

\[
H_3(n) = \left( \prod_{i=1}^{n-2} O_{i+2,2} \left( -\frac{h_{i,1}}{h_{1,2}} \right) \right) H_2(n),
\]

\[
\vdots
\]

\[
H_n(n) = \left( \prod_{i=1}^{1} O_{i+(n-1),n-1} \left( -\frac{h_{i,1}}{h_{1,2}} \right) \right) H_{n-1}(n).
\]

It is obvious that, step by step, the columns are “emptied” until finally the following matrix

\[
H_n(n) = \begin{pmatrix}
    \hat{h}_{0,0} & h_{0,1} & 0 & 0 & 0 & \cdots & 0 \\
    \hat{h}_{1,0} & 0 & h_{1,2} & 0 & 0 & \cdots & 0 \\
    \hat{h}_{2,0} & 0 & 0 & h_{1,2} & 0 & \cdots & 0 \\
    \hat{h}_{3,0} & 0 & 0 & 0 & h_{1,2} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \hat{h}_{n-1,0} & 0 & 0 & 0 & 0 & \cdots & h_{1,2} \\
    \hat{h}_{n,0} & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{(n+1) \times (n+1)}
\]
is obtained, where

\[
\tilde{h}_{i,0} = \begin{cases} 
  h_{0,0}, & \text{if } i = 0; \\
  h_{1,0} - \frac{h_{1,1}}{h_{0,1}} h_{0,0}, & \text{if } i = 1; \\
  h_{i,0} - \frac{h_{i,1}}{h_{0,1}} h_{0,0} - \frac{i - 1}{h_{i,1}} \sum_{k=1}^{i-1} h_{i-k,1} \tilde{h}_{k,0}, & \text{if } i \geq 2.
\end{cases}
\]  

(11)

Evidently, \( d_n = \det(H_n(n)) \). Expanding the determinant along the last row of \( \det(H_n(n)) \), we obtain

\[
d_n = (-1)^n \tilde{h}_{n,0} h_{0,1} (h_{1,2})^{n-1}, \quad (n \geq 1).
\]  

(12)

Finally, after some simplification, it follows that

\[
d_n = (-1)^n \tilde{h}_{n,0} h_{0,1} (h_{1,2})^{n-1}
\]

\[
= (-1)^n h_{0,1} (h_{1,2})^{n-1} \left[ h_{n,0} - \frac{h_{n,1}}{h_{0,1}} h_{0,0} - \frac{n-1}{h_{n,1}} \sum_{k=1}^{n-1} h_{n-k,1} \tilde{h}_{k,0} \right] \quad (\text{by } (11))
\]

\[
= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + (-1)^{n+1} (h_{1,2})^{n-1} h_{n,1} h_{0,0} + (-1)^{n+1} h_{0,1} (h_{1,2})^{n-2} \sum_{k=1}^{n-1} h_{n-k,1} \tilde{h}_{k,0}
\]

\[
= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + (-1)^{n+1} (h_{1,2})^{n-1} h_{n,1} h_{0,0} + \sum_{k=1}^{n-1} h_{n-k,1} (-h_{1,2})^{n-k-1} d_k \quad (\text{by } (12))
\]

\[
= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + \sum_{k=0}^{n-1} h_{n-k,1} (-h_{1,2})^{n-k-1} d_k.
\]

and the result follows.

In Proposition 4, if we take \( h_{0,0} = h_{0,1} = 1, h_{1,2} = 1, h_{i,0} = c \) and \( h_{i,1} = \hat{\mu}_i \) for \( i \geq 1 \), then we obtain

\[
d_n = \begin{cases} 
  1, & \text{if } n = 0; \\
  (-1)^n c + \sum_{k=0}^{n-1} \hat{\mu}_{n-k} (-1)^{n-k-1} d_k, & \text{if } n \geq 1.
\end{cases}
\]

Now, if \((d_n)_{n \geq 0} \in \{F, \mathcal{L}, \mathcal{J}, \mathcal{P}\}\), then

\[
\hat{\mu}_n = c + (-1)^{n-1} d_n + \sum_{k=1}^{n-1} (-1)^{k+1} \hat{\mu}_{n-k} d_k,
\]

from which we determine the sequence \((\hat{\mu}_i)_{i \geq 1}\). Now, we form

\[
H(\infty) = \begin{pmatrix} 
  1 & | & 1 & 0 & \cdots \\
  \vdots & | & \vdots & \ddots & \vdots \\
  c & | & c & \cdots & c \\
  T(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \ldots, (\hat{\mu}_1, 1, 0, 0, \ldots)(\infty)
\end{pmatrix}.
\]

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Finally, the sequences \((\lambda_i)_{i \geq 0}\) and \((\mu_i)_{i \geq 0}\) are determined by the equation \(A(n) = L(n) \cdot H(n) \cdot U(n)\).

4 Acknowledgement

Our special thanks go to the Research Institute for Fundamental Sciences, Tabriz, Iran, for having sponsored this paper.

References


2010 Mathematics Subject Classification: Primary 11C20; Secondary 15B36, 10A35.

Keywords: determinant, generalized Pascal triangle, Toeplitz matrix, matrix factorization, convolution matrix, lower Hessenberg matrix.

(Concerned with sequences [A000032](http://oeis.org/A000032), [A000045](http://oeis.org/A000045), [A000129](http://oeis.org/A000129), and [A001045](http://oeis.org/A001045).)

Received December 3 2013; revised version received March 2 2014. Published in *Journal of Integer Sequences*, March 26 2014.

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