Asymptotic Series for Hofstadter’s Figure-Figure Sequences

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Abstract
We compute asymptotic series for Hofstadter’s figure-figure sequences.

1 Introduction
We consider disjoint partitions of the set of strictly positive integers into two subsets such that one set, $B$, consists of the differences of consecutive elements of the other set, $A$, and a given difference appears at most once. There are many such partitions. We call $a$ the (strictly increasing) sequence enumerating $A$, and $b$ the (injective) sequence of its first differences, both with offset 1. Hofstadter’s figure-figure sequences are the sequences $a$ and $b$ corresponding to the partition with the set $A$ lexicographically minimal. This is equivalent to $b$ being increasing. The sequences read

$$a_n = 1, 3, 7, 12, 18, 26, 35, 45, 56, 69, \ldots$$  \hspace{1cm} \text{(OEIS A005228)},

$$b_n = 2, 4, 5, 6, 8, 9, 10, 11, 13, 14, \ldots$$  \hspace{1cm} \text{(OEIS A030124)}.

These sequences were introduced by Hofstadter in [2, p. 73]. They appear as an example of complementary sequences in [3]. Their asymptotic behavior does not seem to be given

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anywhere in the literature except for the asymptotic equivalents mentioned by Hasler and Wilson in the related OEIS entries [1]. In this article, we compute asymptotic series for these sequences.

We have by definition \( b_n = a_{n+1} - a_n \), so \( a_n = 1 + \sum_{k=1}^{n-1} b_k \). Since the sequence \( a \) is strictly increasing, given any \( n \geq 1 \), there is a unique \( k \geq 1 \) such that \( a_k - k < n \leq a_{k+1} - (k + 1) \). This defines a sequence \( u \) by letting \( u_n \) be this \( k \). Therefore,

\[
a(u_n) - u_n < n \leq a(u_n + 1) - (u_n + 1).
\]

The sequence \( u \) is non-decreasing (actually, \( u_{n+1} - u_n \in \{0, 1\} \)) and \( u_1 = 1 \). It reads

\[ u_n = 1, 2, 2, 2, 3, 3, 3, 4, 4, \ldots \]  
(OEIS \textbf{A225687}).

The partition condition implies

\[ b_n = n + u_n. \]

As a consequence,

\[
a_n = 1 + \frac{(n-1)n}{2} + \sum_{k=1}^{n-1} u_k.
\]

\[ (2) \]

## 2 Bounds and asymptotic equivalents

Since \( u_n \geq 1 \), we have \( a_n \geq \frac{1}{2} n(n + 1) \). Therefore, the left inequality of (1) implies \( \frac{1}{2} u_n (u_n + 1) - u_n \leq n - 1 \), or \( u_n^2 - u_n - 2(n - 1) \leq 0 \), so \( u_n \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2(n - 1)} \), and finally

\[ 1 \leq u_n < \sqrt{2n} + \frac{1}{2}. \]

This implies \( n + 1 \leq b_n < n + \sqrt{2n} + \frac{1}{2} \), so

\[ b_n \sim n. \]

The upper bound on \( u \) implies in turn \( a_n < 1 + \frac{1}{2} (n - 1)n + \sum_{k=1}^{n-1} (\sqrt{2k + \frac{1}{2}}) \). Since the function \( \sqrt{x} \) is strictly increasing, we have \( \sum_{k=1}^{n-1} \sqrt{k} < \int_1^n \sqrt{x} \, dx = \frac{2}{3} (n^{3/2} - 1) \). Therefore

\[ \frac{n^2}{2} + \frac{n}{2} \leq a_n < \frac{n^2}{2} + \frac{2^{3/2}}{3} n^{3/2} - \frac{1}{3} \]

and in particular

\[ a_n \sim \frac{n^2}{2}. \]

The relation \( a_n < \frac{n^2}{2} + \frac{2^{3/2}}{3} (\frac{n}{2})^{3/2} - \frac{1}{3} \) and the right inequality of (1) imply \( n < \frac{(u_n + 1)^2}{2} + \frac{2^{3/2}}{3} (u_n + 1)^{3/2} - u_n - \frac{4}{3} \), which implies \( u_n \to +\infty \). Therefore \( 2n \leq u_n^2 + O(u_n^{3/2}) \), but we saw that \( u_n = O(\sqrt{n}) \), so \( O(u_n^{3/2}) \subseteq O(n^{3/4}) \subseteq o(n) \), so \( u_n^2 \geq 2n + o(n) \), so \( u_n \geq \sqrt{2n} + o(\sqrt{n}) \).

Combining this with the above upper bound, we obtain

\[ u_n \sim \sqrt{2n} \]

and in particular \( O(u_n) = O(\sqrt{n}) \).
3 Asymptotic series

Since \( a_n \sim \frac{n^2}{2} \), we have \( a_{n+1} - a_n = O(n) \). Now (1) gives \( a(u_n) = n + O(u_n) \). On the other hand, (2) gives \( a_n = \frac{n^2}{2} + \sum_{k=1}^{n-1} u_k + O(n) \), therefore \( \frac{u_n^2}{2} + \sum_{k=1}^{u_n-1} u_k = n + O(u_n) \). Since \( u_n = O(\sqrt{n}) \), we can increment the upper limit of the summation index by 1, and since \( O(u_n) = O(\sqrt{n}) \), we obtain the main relation

\[
\frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k = n + O(\sqrt{n}).
\]

We are now ready to prove by induction that for all \( K \geq 1 \), we have the asymptotic expansion

\[
u_n = \sum_{k=1}^{K} (-1)^{k+1} \frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2^j + 1)} \left( \frac{n}{2} \right)^{1/2^k} + o\left(n^{1/2^k}\right).
\]

Indeed, the case \( K = 1 \) reduces to \( u_n \sim \sqrt{2n} \), which we already proved. We also prove the case \( K = 2 \) separately since it is slightly different from the general case. We write \( u_n = \sqrt{2n} + v_n \) with \( v_n = o(\sqrt{n}) \). We have

\[
\frac{u_n^2}{2} - n = \sqrt{2n} v_n + \frac{v_n^2}{2}.
\]

We do not know \textit{a priori} that \( v_n^2 = O(\sqrt{n}) \), and that is why we have to prove this case separately. We also have

\[
\sum_{k=1}^{u_n} u_k = \sqrt{2} \sum_{k=1}^{u_n} \sqrt{k} + \sum_{k=1}^{u_n} v_k = \frac{2^{3/2}}{3} u_n^{3/2} + o\left(O(u_n)^{3/2}\right) + \sum_{k=1}^{u_n} v_k.
\]

We have \( \sum_{k=1}^{u_n} v_k = o\left(O(\sqrt{n})^{3/2}\right) \subseteq o(n^{3/4}) \) and \( o\left(O(u_n)^{3/2}\right) \subseteq o(n^{3/4}) \). We also have \( u_n^{3/2} \sim (\sqrt{2n})^{3/2} = (2n)^{3/4} \). Therefore,

\[
\frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k - n = \sqrt{2n} v_n + \frac{v_n^2}{2} + \frac{2^{9/4}}{3} n^{3/4} + o(n^{3/4}).
\]

This has to be \( O(\sqrt{n}) \) by the main relation. Dividing the right-hand side by \( \sqrt{2n} \), we obtain

\[
v_n + \frac{v_n^2}{2\sqrt{2n}} + \frac{2^{7/4}}{3} n^{1/4} = o(n^{1/4}).
\]

Since \( v_n = o(\sqrt{n}) \), we have \( \frac{v_n^2}{2\sqrt{2n}} = o(v_n) \), so

\[
v_n + \frac{2^{7/4}}{3} n^{1/4} = o(n^{1/4}) + o(v_n),
\]
so \( v_n \sim -\frac{2^2}{3} \left( \frac{n}{2} \right)^{1/4} \), as desired.

Now, suppose that the expansion holds for some \( K \geq 2 \). We prove it for \( K + 1 \). It will be convenient to denote the coefficients of the expansion by

\[
\alpha_k = (-1)^{k+1} \frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2j + 1)}.
\]

so \( \alpha_1 = 2 \). We write \( v_n = o(n^{1/2K}) \) for the remainder in (3). Then (3) gives

\[
u_n^2 = \left( \sqrt{2n} + \sum_{k=2}^{K} \alpha_k \left( \frac{n}{2} \right)^{1/2k} + v_n \right)^2 = 2n \left( 1 + \frac{1}{\sqrt{2n}} \sum_{k=2}^{K} \alpha_k \left( \frac{n}{2} \right)^{1/2k} + \frac{v_n}{\sqrt{2n}} \right)^2
\]

\[
= 2n \left( 1 + \frac{2}{\sqrt{2n}} \sum_{k=2}^{K} \alpha_k \left( \frac{n}{2} \right)^{1/2k} + 2 \frac{v_n}{\sqrt{2n}} + O(n^{1/4+1/4-1}) + O \left( \frac{v_n^2}{n} \right) + O(n^{1/4-1} v_n) \right).
\]

Since \( K \geq 2 \), we have \( v_n = o \left( n^{1/2K} \right) \subseteq o \left( n^{1/4} \right) \). Therefore

\[
\frac{v_n^2}{2} - n = 2 \sum_{k=2}^{K} \alpha_k \left( \frac{n}{2} \right)^{1/2+1/2k} + \sqrt{2n} v_n + O(n).
\]

On the other hand,

\[
\sum_{k=1}^{u_n} u_k = \sum_{k=1}^{K} 2^{k+1} - 2k + 1 \alpha_k \left( \frac{u_n}{2} \right)^{1+1/2k} + o \left( u_n^{1+1/2K} \right)
\]

\[
= \sum_{k=1}^{K} 2^{k+1} - 2k + 1 \alpha_k \left( \frac{n}{2} \right)^{1/2+1/2k+1} + o \left( n^{1/2+1/2K+1} \right).
\]

Therefore

\[
\frac{u_n^2}{2} + \sum_{k=1}^{u_n} u_k - n = 2 \sum_{k=2}^{K} \alpha_k \left( \frac{n}{2} \right)^{1/2+1/2k} + \sum_{k=1}^{K} \alpha_k \frac{2^{k+1}}{2k + 1} \left( \frac{n}{2} \right)^{1/2+1/2k+1}
\]

\[
+ \sqrt{2n} v_n + o \left( n^{1/2+1/2K+1} \right)
\]

\[
= \alpha K \frac{2^{K+1}}{2K + 1} \left( \frac{n}{2} \right)^{1/2+1/2K+1} + 2 \left( \frac{n}{2} \right)^{1/2} v_n + o \left( n^{1/2+1/2K+1} \right)
\]

since the terms in the sums cancel out except for the last in the second sum. This expression has to be \( O(\sqrt{n}) \) by the main relation, so \( v_n \sim -\frac{2^K}{2^{K+1} \alpha K \left( \frac{n}{2} \right)^{1/2K+1}} \), as desired.

From the expansion of \( u_n \), we find that of \( b_n = n + u_n \), and that of \( a_n \) by term-by-term integration. We obtain

\[
b_n = n + \sum_{k=1}^{K} (-1)^{k+1} \frac{2^{1+(k-1)k/2}}{\prod_{j=1}^{k-1}(2j + 1)} \left( \frac{n}{2} \right)^{1/2k} + o \left( n^{1/2K} \right)
\]
and
\[ a_n = \frac{n^2}{2} + \sum_{k=1}^{K} (-1)^{k+1} \frac{2^{k(k+1)/2}}{\prod_{j=1}^{k} (2^j + 1)} \left( \frac{n}{2} \right)^{1+1/2^k} + o\left( n^{1+1/2^K} \right). \]

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