\textbf{\textit{n}}-\textbf{Color Odd Self-Inverse Compositions}

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\textbf{Abstract}

An \textit{n}-color odd self-inverse composition is an \textit{n}-color self-inverse composition with odd parts. In this paper, we obtain generating functions, explicit formulas, and recurrence formulas for \textit{n}-color odd self-inverse compositions.

\section{Introduction}

In the classical theory of partitions, compositions were first defined by MacMahon [9] as ordered partitions. For example, there are 5 partitions and 8 compositions of 4. The partitions are 4, 31, 22, 211, 1111 and the compositions are 4, 31, 13, 22, 211, 121, 112, 1111.

Agarwal and Andrews [1] defined an \textit{n}-color partition as a partition in which a part of size \textit{n} can come in \textit{n} different colors. They denoted different colors by subscripts: \textit{n}_1, \textit{n}_2, \ldots, \textit{n}_n. In analogy with MacMahon’s ordinary compositions, Agarwal [2] defined an \textit{n}-color composition as an \textit{n}-color ordered partition. Thus, for example, there are 8 \textit{n}-color compositions of 3, viz.,

$$3_1, 3_2, 3_3, 2_11_1, 2_21_1, 1_12_1, 1_12_2, 1_11_11_1.$$  

More properties of \textit{n}-color compositions were given in [3, 5].

\textbf{Definition 1.} ([9]) A composition is said to be self-inverse when the parts of the composition read from left to right are identical with the parts when read from right to left.

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In analogy with the definition above for classical self-inverse compositions, Narang and Agarwal [10] defined an \( n \)-color self-inverse composition and gave some properties of them.

**Definition 2.** ([10]) An \( n \)-color odd composition is an \( n \)-color composition with odd parts.

For example there are 8 \( n \)-color self-inverse compositions of 4, viz.,

\[
4_1, 4_2, 4_3, 4_4, 2_12_1, 2_22_2, 1_12_11_1, 1_12_21_1.
\]

In 2010, the author [6] also defined an \( n \)-color even self-inverse composition and gave some properties.

**Definition 3.** ([6]) An \( n \)-color even composition is an \( n \)-color composition whose parts are even.

**Definition 4.** ([6]) An \( n \)-color even composition whose parts read from left to right are identical with when read from right to left is called an \( n \)-color even self-inverse composition.

Thus, for example, there are 8 \( n \)-color even self-inverse compositions of 4, viz.,

\[
4_1, 4_2, 4_3, 4_4, 2_12_1, 2_22_2, 1_12_21_1, 2_22_2.
\]

And there are 6 \( n \)-color even self-inverse compositions of 4, viz.,

\[
4_1, 4_2, 4_3, 4_4, 2_12_1, 2_22_2.
\]

Recently, the author [7] studied \( n \)-color odd compositions.

**Definition 5.** ([7]) An \( n \)-color odd composition is an \( n \)-color composition whose parts are odd.

Thus, for example, there are 7 \( n \)-color odd compositions of 4, viz.,

\[
3_11_1, 3_21_1, 3_31_1, 1_13_1, 1_13_2, 1_13_3, 1_11_11_11_1.
\]

In this paper, we shall study \( n \)-color odd self-inverse compositions.

**Definition 6.** An \( n \)-color odd composition whose parts read from left to right are identical with when read from right to left is called an \( n \)-color odd self-inverse composition.

Thus, for example, there are 4 \( n \)-color odd self-inverse compositions of 6, viz.,

\[
3_13_1, 3_23_2, 3_33_3, 1_11_11_11_11_1.
\]

In section 2 we shall give explicit formulas, recurrence formulas, generating functions for \( n \)-color odd self-inverse compositions.

The author [7] proved the following theorems.
Theorem 7. ([7]) Let $C_o(m, q)$ and $C_o(q)$ denote the enumerative generating functions for $C_o(m, \nu)$ and $C_o(\nu)$, respectively, where $C_o(m, \nu)$ is the number of $n$-color odd compositions of $\nu$ into $m$ parts and $C_o(\nu)$ is the number of $n$-color odd compositions of $\nu$. Then

$$C_o(m, q) = \frac{q^m (1 + q^2)^m}{(1 - q^2)^{2m}}, \quad (1)$$

$$C_o(q) = \frac{q + q^3}{1 - q - 2q^2 - q^3 + q^4}, \quad (2)$$

$$C_o(m, \nu) = \sum_{i+j=\frac{\nu-m}{2} m} \binom{2m + i - 1}{2m - 1} \binom{m}{j}, \quad (3)$$

$$C_o(\nu) = \sum_{m \leq \nu} \sum_{i+j=\frac{\nu-m}{2} m} \binom{2m + i - 1}{2m - 1} \binom{m}{j}. \quad (4)$$

where $(\nu - m)$ is even, and $(\nu - m) \geq 0; 0 \leq i, j$ are integers.

Theorem 8. ([7]) Let $C_o(\nu)$ denote the number of $n$-color odd compositions of $\nu$. Then

$$C_o(1) = 1, C_o(2) = 1, C_o(3) = 4, C_o(4) = 7 \quad \text{and} \quad C_o(\nu) = C_o(\nu - 1) + 2C_o(\nu - 2) + C_o(\nu - 3) - C_o(\nu - 4) \quad \text{for} \ \nu > 4.$$

2 Main results

In this section, we first prove the following explicit formulas for the number of $n$-color odd self-inverse compositions.

Theorem 9. Let $S(O, \nu)$ denote the number of $n$-color odd self-inverse compositions of $\nu$. Then

$$(1) \quad S(O, 2\nu + 1) = (2\nu + 1) + \sum_{t=1}^{2\nu-1} \sum_{m \leq \frac{2n+1-t}{2}} \sum_{i+j=\frac{2n+1-t-m}{2}} t \binom{2m + i - 1}{2m - 1} \binom{m}{j},$$

where $\nu = 0, 1, 2, \ldots; \quad t = 2k + 1, k = 0, 1, 2, \ldots, (\nu - 1); \quad 0 \leq \frac{2\nu+1-t-2m}{2} \quad \text{is even}; \quad 0 \leq i, j$ are integers.

$$(2) \quad S(O, 2\nu) = \sum_{m \leq \nu} \sum_{i+j=\frac{\nu-m}{2} m} \binom{2m + i - 1}{2m - 1} \binom{m}{j},$$

where $0 \leq \nu - m$ is even, and $0 \leq i, j$ are integers.
Proof. (1) Obviously, an odd number which is \(2\nu + 1\) \((\nu = 0, 1, 2, \ldots)\) can have odd self-inverse \(n\)-color compositions only when the number of parts is odd. There are \(2\nu + 1\) \(n\)-color odd self-inverse compositions when the number of parts is only one. An odd self-inverse compositions of \(2\nu + 1\) into \(2m + 1\) \((m \geq 1)\) parts can be read as a central part, say, \(t\) \((\text{where } t \text{ is odd})\) and two identical odd \(n\)-color compositions of \(\frac{2\nu + 1 - t}{2}\) into \(m\) parts on each side of the central part. The number of odd \(n\)-color compositions of \(\frac{2\nu + 1 - t}{2}\) into \(m\) parts is \(C_o(m, \frac{2\nu + 1 - t}{2})\) by equation (3). Now the central part can appear in \(t\) ways. Therefore, the number of \(n\)-color odd self-inverse compositions of \(2\nu + 1\) is

\[
S(O, 2\nu + 1) = (2\nu + 1) + \sum_{t=1}^{2\nu-1} \sum_{m \leq \frac{2\nu + 1 - t}{2}} tC_o\left(m, \frac{2\nu + 1 - t}{2}\right)
\]

\[
= (2\nu + 1) + \sum_{t=1}^{2\nu-1} \sum_{m \leq \frac{2\nu + 1 - t}{2}} \sum_{i+j=\frac{2\nu + 1 - t}{2}} t\left(\frac{2m + i - 1}{2m - 1}\right)\left(m\right)\left(j\right).
\]

(2) For even numbers \(2\nu\) \((\nu = 1, 2, \ldots)\), we can have odd self-inverse \(n\)-color compositions only when the number of parts is even, and the two identical odd \(n\)-color compositions are exactly odd \(n\)-color compositions of \(\nu\), from equation (4) we see that the number of these is

\[
\sum_{m \leq \nu} \sum_{i+j=\frac{\nu-m}{2}} \left(\frac{2m + i - 1}{2m - 1}\right)\left(m\right)\left(j\right).
\]

Hence, the number of \(n\)-color odd self-inverse compositions of \(2\nu\) is

\[
S(O, 2\nu) = \sum_{m \leq \nu} \sum_{i+j=\frac{\nu-m}{2}} \left(\frac{2m + i - 1}{2m - 1}\right)\left(m\right)\left(j\right).
\]

We complete the proof of this theorem. \(\square\)

From the proof of this theorem we can see that odd \(n\) have \(n\)-color odd self-inverse compositions where the number of parts is odd. And even \(n\) have \(n\)-color odd self-inverse compositions where the number of parts is even. Let \(S_o(\nu, m)\) denote the number of \(n\)-color odd self-inverse compositions of \(\nu\) into \(m\) parts. Then we can get the following formula easily:

\[
S_o(2k + 1, 2l + 1) = \sum_{t=1}^{2k-1} \sum_{i+j=\frac{2k+1-t-2l}{4}} \left(\frac{2l + i - 1}{2l - 1}\right)\left(l\right)\left(j\right).
\]

where \(t\) is odd, \(k, l\) are integers and \(k, l \geq 0\).

\[
S_o(2k, 2l) = \sum_{i+j=\frac{k-l}{2}} \left(\frac{2l + i - 1}{2l - 1}\right)\left(l\right)\left(j\right).
\]
Table 1: $S_{o}(\nu, m)$ when both $\nu$ and $m$ are odd

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<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
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where $k, l$ are integers and $k, l \geq 0$.

Now $S_{o}(\nu, m)$ with $\nu, m = 1, 2, \ldots, 20$ is given in Tables 1 and 2.

From Tables 1 and 2 we can see the recurrence formulas for the number of the $n$-color odd self-inverse compositions of $\nu$. So we prove the following recurrence relations.

Table 2: $S_{o}(\nu, m)$ when both $\nu$ and $m$ are even

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<th>6</th>
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<th>10</th>
<th>12</th>
<th>14</th>
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**Theorem 10.** Let $s_{\nu}$ and $t_{\nu}$ denote the number of $n$-color odd self-inverse compositions for $2\nu + 1$ and $2\nu$, respectively. Then

(1) \[ s_0 = 1, \; s_1 = 4, \; s_2 = 9, \; s_3 = 19 \text{ and} \]
\[ s_{\nu} = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4} \text{ for } \nu > 3 \]

(2) \[ t_1 = 1, \; t_2 = 1, \; t_3 = 4, \; t_4 = 7 \text{ and} \]
\[ t_{\nu} = t_{\nu-1} + 2t_{\nu-2} + t_{\nu-3} - t_{\nu-4} \text{ for } \nu > 4. \]
Proof. (Combinatorial) (1) To prove that $s_\nu = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4}$, we split the \(n\)-color odd self-inverse compositions enumerated by $s_\nu + s_{\nu-4}$ into four classes:

(A) $s_\nu$ with 1\(_1\) on both ends.
(B) $s_\nu$ with 3\(_3\) on both ends.
(C) $s_\nu$ with $h_t$ on both ends, $h > 1$, $1 \leq t \leq h - 2$ and \(n\)-color odd self-inverse compositions of $2\nu + 1$ of form $(2\nu + 1)_u$, $1 \leq u \leq 2\nu - 3$.
(D) $s_\nu$ with $h_t$ on both ends except 3\(_3\), $h > 1$, $h - 1 \leq t \leq h$, $(2\nu + 1)_u$, $2\nu - 2 \leq u \leq 2\nu + 1$ and those enumerated by $s_{\nu-4}$.

We transform the \(n\)-color odd self-inverse compositions in class (A) by deleting 1\(_1\) on both ends. This produces \(n\)-color odd self-inverse compositions enumerated by $s_{\nu-1}$. Conversely, for any \(n\)-color odd composition enumerated by $s_{\nu-1}$ we add 1\(_1\) on both ends to produce the elements of the class (A). In this way we establish that there are exactly $s_{\nu-1}$ elements in the class (A).

Similarly, we can produce $s_{\nu-3}$ \(n\)-color odd self-inverse compositions in class (B) by deleting 3\(_3\) on both ends.

Next, we transform the \(n\)-color odd self-inverse compositions in class (C) by subtracting 2 from $h$, that is, replacing $h_t$ by $(h - 2)_t$ and subtracting 4 from $2\nu + 1$ of $(2\nu + 1)_u$, $1 \leq u \leq 2\nu - 3$. This transformation also establishes the fact that there are exactly $s_{\nu-2}$ elements in class (C).

Finally, we transform the elements in class (D) as follows: Subtract 2 from $h_t$ on both ends, that is, replace $h_t$ by $(h - 2)_{(t-2)}$, $h > 3$, $h - 1 \leq t \leq h$, while replace $h_t$ by $(h - 2)_{(t-1)}$ when $h = 3$, $t = 2$. We will get those \(n\)-color odd self-inverse compositions of $2\nu - 3$ with $h_t$ on both ends, $h - 1 \leq t \leq h$ except self-inverse odd compositions in one part. We also replace $(2\nu + 1)_u$ by $(2\nu - 3)_{u-4}$, $2\nu - 2 \leq u \leq 2\nu + 1$. To get the remaining \(n\)-color odd compositions from $s_{\nu-4}$ we add 2 to both ends, that is, replace $h_t$ by $(h + 2)_t$. For \(n\)-color odd self-inverse compositions into one part we add 4, that is, replace $(2\nu - 7)_t$ by $(2\nu - 3)_t$, $1 \leq t \leq 2\nu - 7$. We see that the number of \(n\)-color odd self-inverse compositions in class (D) is also equal to $s_{\nu-2}$. Hence, we have $s_\nu + s_{\nu-4} = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3}$. viz., $s_\nu = s_{\nu-1} + 2s_{\nu-2} + s_{\nu-3} - s_{\nu-4}$.

(2) From Theorem 8 and Theorem 9, we obtain the recurrence formula of $t_\nu$ easily. Thus, we complete the proof.

We easily get the following generating functions by the recurrence relations.

**Corollary 11.**

\[
\begin{align*}
(1) \quad \sum_{\nu=0}^{\infty} s_\nu q^\nu &= \frac{(1 + q)^3}{1 - q - 2q^2 - q^3 + q^4}, \\
(2) \quad \sum_{\nu=1}^{\infty} t_\nu q^\nu &= \frac{q + q^3}{1 - q - 2q^2 - q^3 + q^4}.
\end{align*}
\]
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References


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