A Simplified Binet Formula for 
k-\textit{Generalized Fibonacci Numbers}

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Abstract  
In this paper, we present a Binet-style formula that can be used to produce the \( k \)-generalized Fibonacci numbers (that is, the Tribonacci, Tetranacci, etc.). Furthermore, we show that in fact one needs only take the integer closest to the first term of this Binet-style formula in order to generate the desired sequence.

1 Introduction  
Let \( k \geq 2 \) and define \( F^{(k)}_n \), the \( n \)-th \( k \)-generalized Fibonacci number, as follows:

\[
F^{(k)}_n = \begin{cases} 
0, & \text{if } n < 1; \\
1, & \text{if } n = 1; \\
F^{(k)}_{n-1} + F^{(k)}_{n-2} + \cdots + F^{(k)}_{n-k}, & \text{if } n > 1.
\end{cases}
\]
These numbers are also called generalized Fibonacci numbers of order \( k \), Fibonacci \( k \)-step numbers, Fibonacci \( k \)-sequences, or \( k \)-bonacci numbers. Note that for \( k = 2 \), we have \( F^{(2)}_n = F_n \), our familiar Fibonacci numbers. For \( k = 3 \) we have the so-called Tribonacci (sequence number A000073 in Sloane's Encyclopedia of Integer Sequences), followed by the Tetranacci (A000078) for \( k = 4 \), and so on. According to Kessler and Schiff [6], these numbers also appear in probability theory and in certain sorting algorithms. We present here a chart of these numbers for the first few values of \( k \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>name</th>
<th>first few non-zero terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Fibonacci</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, . . .</td>
</tr>
<tr>
<td>3</td>
<td>Tribonacci</td>
<td>1, 1, 2, 4, 7, 13, 24, 44, 81, . . .</td>
</tr>
<tr>
<td>4</td>
<td>Tetranacci</td>
<td>1, 1, 2, 4, 8, 15, 29, 56, 108, . . .</td>
</tr>
<tr>
<td>5</td>
<td>Pentanacci</td>
<td>1, 1, 2, 4, 8, 16, 31, 61, 120, . . .</td>
</tr>
</tbody>
</table>

We remind the reader of the famous Binet formula (also known as the de Moivre formula) that can be used to calculate \( F_n \), the Fibonacci numbers:

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] = \frac{\alpha^n - \beta^n}{\alpha - \beta}
\]

for \( \alpha > \beta \) the two roots of \( x^2 - x - 1 = 0 \). For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

\[
F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)\alpha^{n-1}} + \frac{\beta - 1}{2 + 3(\beta - 2)\beta^{n-1}}
\]

for \( \alpha > \beta \) the two roots of \( x^2 - x - 1 = 0 \). For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

\[
F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)\alpha^{n-1}} + \frac{\beta - 1}{2 + 3(\beta - 2)\beta^{n-1}}
\]

We leave the details to the reader.

Our first (and very minor) result is the following representation of \( F^{(k)}_n \):

**Theorem 1.** For \( F^{(k)}_n \) the \( n^{th} \) \( k \)-generalized Fibonacci number, then

\[
F^{(k)}_n = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}
\]

for \( \alpha_1, \ldots, \alpha_k \) the roots of \( x^k - x^{k-1} - \cdots - 1 = 0 \).

This is a new presentation, but hardly a new result. There are many other ways of representing these \( k \)-generalized Fibonacci numbers, as seen in the articles [2, 3, 4, 5, 7, 8, 9]. Our Eq. (2) of Theorem 1 is perhaps slightly easier to understand, and it also allows us to do
some analysis (as seen below). We point out that for $k = 2$, Eq. (2) reduces to the variant of the Binet formula (for the standard Fibonacci numbers) from Eq. (1).

As shown in three distinct proofs [9, 10, 13], the equation $x^k - x^{k-1} - \cdots - 1 = 0$ from Theorem 1 has just one root $\alpha$ such that $|\alpha| > 1$, and the other roots are strictly inside the unit circle. We can conclude that the contribution of the other roots in Eq. 2 will quickly become trivial, and thus:

$$F_n^{(k)} \approx \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \quad \text{for } n \text{ sufficiently large.} \quad (3)$$

It’s well known that for the Fibonacci sequence $F_n^{(2)} = F_n$, the “sufficiently large” $n$ in Eq. (3) is $n = 0$, as shown here:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_n^{(6)}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>$\frac{\alpha - 1}{2 + 7(\alpha - 2)} \alpha^5$</td>
<td>0.263</td>
<td>0.522</td>
<td>1.035</td>
<td>2.053</td>
<td>4.072</td>
<td>8.078</td>
<td>16.023</td>
</tr>
<tr>
<td></td>
<td>31.782</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.218</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is perhaps surprising to discover that a similar statement holds for all the $k$-generalized Fibonacci numbers. Let’s first define $\text{rnd}(x)$ to be the the value of $x$ rounded to the nearest integer: $\text{rnd}(x) = \lfloor x + \frac{1}{2} \rfloor$. Then, our main result is the following:

**Theorem 2.** For $F_n^{(k)}$ the $n^{th}$ $k$-generalized Fibonacci number, then

$$F_n^{(k)} = \text{rnd} \left( \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right)$$

for all $n \geq 2 - k$ and for $\alpha$ the unique positive root of $x^k - x^{k-1} - \cdots - 1 = 0$.

We point out that this theorem is not as trivial as one might think. Note the error term for the generalized Fibonacci numbers of order $k = 6$, as seen in the following chart; it is not monotone decreasing in absolute value.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
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<td>$F_n^{(6)}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
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<td>8.078</td>
<td>16.023</td>
<td>31.782</td>
</tr>
<tr>
<td></td>
<td>.218</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We also point out that not every recurrence sequence admits such a simple formula as seen in Theorem 2. Consider, for example, the scaled Fibonacci sequence 10, 10, 20, 30, 50, 80, . . . , which has Binet formula:

$$\frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{10}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$
This can be written as \( \text{rnd} \left( \frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \right) \), but only for \( n \geq 5 \). As another example, the sequence 1, 2, 8, 24, 80, \ldots \) (defined by \( G_n = 2G_{n-1} + 4G_{n-2} \)) can be written as

\[
G_n = \frac{(1 + \sqrt{5})^n}{2\sqrt{5}} - \frac{(1 - \sqrt{5})^n}{2\sqrt{5}},
\]

but because both \( 1 + \sqrt{5} \) and \( 1 - \sqrt{5} \) have absolute value greater than 1, then it would be impossible to express \( G_n \) in terms of just one of these two numbers.

## 2 Previous Results

We point out that for \( k = 3 \) (the Tribonacci numbers), our Theorem 2 was found earlier by Spickerman [11]. His formula (modified slightly to match our notation) reads as follows, where \( \alpha \) is the real root, and \( \sigma \) and \( \bar{\sigma} \) are the two complex roots, of \( x^3 - x^2 - x - 1 = 0 \):

\[
F_n^{(3)} = \text{rnd} \left( \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \bar{\sigma})} \alpha^{n-1} \right) \tag{4}
\]

It is not hard to show that for \( k = 3 \), our coefficient \( \frac{\alpha^{-1}}{x+(k+1)(\alpha-2)} \) from Theorem 2 is equal to Spickerman’s coefficient \( \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \bar{\sigma})} \). We leave the details to the reader.

In a subsequent article [12], Spickerman and Joyner developed a more complex version of our Theorem 1 to represent the generalized Fibonacci numbers. Using our notation, and with \( \{\alpha_i\} \) the set of roots of \( x^k - x^{k-1} - \cdots - 1 = 0 \), their formula reads

\[
F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{-1}}{2\alpha_i^{k} - (k + 1)} \alpha_i^{n-1} \tag{5}
\]

It is surprising that even after calculating out the appropriate constants in their Eq. (5) for \( 2 \leq k \leq 10 \), neither Spickerman nor Joyner noted that they could have simply taken the first term in Eq. (5) for all \( n \geq 0 \), as Spickerman did in Eq. (4) for \( k = 3 \).

The Spickerman-Joyner Eq. (5) was extended by Wolfram [13] to the case with arbitrary starting conditions (rather than the initial sequence 0, 0, \ldots, 0, 1). In the next section we will show that our Eq. (2) in Theorem 1 is equivalent to the Spickerman-Joyner formula given above (and thus is a special case of Wolfram’s formula).

Finally, we note that the polynomials \( x^k - x^{k-1} - \cdots - 1 \) in Theorem 1 have been studied rather extensively. They are irreducible polynomials with just one zero outside the unit circle. That single zero is located between \( 2(1 - 2^{-k}) \) and 2 (as seen in Wolfram’s article [13]; Miles [9] gave earlier and less precise results). It is also known [13, Lemma 3.11] that the polynomials have Galois group \( S_k \) for \( k \leq 11 \); in particular, their zeros can not be expressed in radicals for \( 5 \leq k \leq 11 \). Wolfram conjectured that the Galois group is always \( S_k \). Cipu and Luca [1] were able to show that the Galois group is not contained in the alternating group \( A_k \), and for \( k \geq 3 \) it is not 2-nilpotent. They point out that this means the zeros of the polynomials \( x^k - x^{k-1} - \cdots - 1 \) for \( k \geq 3 \) can not be constructed by ruler and compass, but the question of whether they are expressible using radicals remains open for \( k \geq 12 \).
3 Preliminary Lemmas

First, a few statements about the number \( \alpha \).

**Lemma 3.** Let \( \alpha > 1 \) be the real positive root of \( x^k - x^{k-1} - \cdots - x - 1 = 0 \). Then,

\[
2 - \frac{1}{k} < \alpha < 2
\]  

(6)

In addition,

\[
2 - \frac{1}{3k} < \alpha < 2 \quad \text{for } k \geq 4.
\]  

(7)

**Proof.** We begin by computing the following chart for \( k \leq 5 \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 2 - \frac{1}{k} )</th>
<th>( 2 - \frac{1}{3k} )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.50</td>
<td>1.833\ldots</td>
<td>1.618\ldots</td>
</tr>
<tr>
<td>3</td>
<td>1.666\ldots</td>
<td>1.889\ldots</td>
<td>1.839\ldots</td>
</tr>
<tr>
<td>4</td>
<td>1.750</td>
<td>1.916\ldots</td>
<td>1.928\ldots</td>
</tr>
<tr>
<td>5</td>
<td>1.800</td>
<td>1.933\ldots</td>
<td>1.966\ldots</td>
</tr>
</tbody>
</table>

It’s clear that \( 2 - \frac{1}{k} < \alpha < 2 \) for \( 2 \leq k \leq 5 \) and that \( 2 - \frac{1}{3k} < \alpha < 2 \) for \( 4 \leq k \leq 5 \). We now focus on \( k \geq 6 \). At this point, we could finish the proof by appealing to \( 2(1 - 2^{-k}) < \alpha < 2 \) as seen in the article [13, Lemma 3.6], but here we present a simpler proof.

Let \( f(x) = (x - 1)(x^k - x^{k-1} - \cdots - x - 1) = x^{k+1} - 2x^k + 1 \). We know from our earlier discussion that \( f(x) \) has one real zero \( \alpha > 1 \). Writing \( f(x) \) as \( x^k(x - 2) + 1 \), we have

\[
f \left( 2 - \frac{1}{3k} \right) = \left(2 - \frac{1}{3k}\right)^k \left(-\frac{1}{3k}\right) + 1
\]  

(8)

For \( k \geq 6 \), it’s easy to show

\[
3k < \left(\frac{5}{3}\right)^k = \left(2 - \frac{1}{3}\right)^k < \left(2 - \frac{1}{3k}\right)^k
\]

Substituting this inequality into the right-hand side of (8), we can re-write (8) as

\[
f \left(2 - \frac{1}{3k}\right) < (3k) \cdot \left(-\frac{1}{3k}\right) + 1 = 0.
\]

Finally, we note that

\[
f(2) = 2^{k+1} - 2 \cdot 2^k + 1 = 1 > 0,
\]

so we can conclude that our root \( \alpha \) is within the desired bounds of \( 2 - 1/3k \) and \( 2 \) for \( k \geq 6 \). \( \square \)

We now have a lemma about the coefficients of \( \alpha^{n-1} \) in Theorems 1 and 2.
Lemma 4. Let $k \geq 2$ be an integer, and let $m^{(k)}(x) = \frac{x - 1}{2 + (k + 1)(x - 2)}$. Then,

1. $m^{(k)}(2 - 1/k) = 1$.
2. $m^{(k)}(2) = \frac{1}{2}$.
3. $m^{(k)}(x)$ is continuous and decreasing on the interval $[2 - 1/k, \infty)$.
4. $m^{(k)}(x) > \frac{1}{x}$ on the interval $(2 - 1/k, 2)$.

Proof. Parts 1 and 2 are immediate. As for 3, note that we can rewrite $m^{(k)}(x)$ as

$$m^{(k)}(x) = \frac{1}{k+1} \left( 1 + \frac{1 - \frac{2}{k+1}}{x - \left(2 - \frac{2}{k+1}\right)} \right)$$

which is simply a scaled translation of the map $y = 1/x$. In particular, since this $m^{(k)}(x)$ has a vertical asymptote at $x = 2 - \frac{2}{k+1}$, then by parts 1 and 2 we can conclude that $m^{(k)}(x)$ is indeed continuous and decreasing on the desired interval.

To show part 4, we first note that in solving $\frac{1}{x} = m^{(k)}(x)$, we obtain a quadratic equation with the two intersection points $x = 2$ and $x = k$. It’s easy to show that $\frac{1}{x} < m^{(k)}(x)$ at $x = 2 - 1/k$, and since both functions $\frac{1}{x}$ and $m^{(k)}(x)$ are continuous on the interval $[2 - 1/k, \infty)$ and intersect only at $x = 2$ and $x = k \geq 2$, we can conclude that $\frac{1}{x} < m^{(k)}(x)$ on the desired interval.

Lemma 5. For a fixed value of $k \geq 2$ and for $n \geq 2 - k$, define $E_n$ to be the error in our Binet approximation of Theorem 2, as follows:

$$E_n = F_n^{(k)} - \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \cdot \alpha^{n-1} = F_n^{(k)} - m^{(k)}(\alpha) \cdot \alpha^{n-1},$$

for $\alpha$ the positive real root of $x^k - x^{k-1} - \cdots - x - 1 = 0$ and $m^{(k)}$ as defined in Lemma 4. Then, $E_n$ satisfies the same recurrence relation as $F_n^{(k)}$:

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad (n \geq 2).$$

Proof. By definition, we know that $F_n^{(k)}$ satisfies the recurrence relation:

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad (9)$$

As for the term $m^{(k)}(\alpha) \cdot \alpha^{n-1}$, note that $\alpha$ is a root of $x^k - x^{k-1} - \cdots - 1 = 0$, which means that $\alpha^k = \alpha^{k-1} + \cdots + 1$, which implies

$$m^{(k)}(\alpha) \cdot \alpha^{n-1} = m^{(k)}(\alpha)\alpha^{n-2} + \cdots + m^{(k)}(\alpha)\alpha^{n-(k+1)} \quad (10)$$

We combine Equations (9) and (10) to obtain the desired result. \qed
4 Proof of Theorem 1

As mentioned above, Spickerman and Joyner [12] proved the following formula for the \(k\)-generalized Fibonacci numbers:

\[ F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)\alpha_i^{n-1}} \] (11)

Recall that the set \(\{\alpha_i\}\) is the set of roots of \(x^k - x^{k-1} - \cdots - 1 = 0\). We now show that this formula is equivalent to our Eq. (2) in Theorem 1:

\[ F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)}\alpha_i^{n-1} \] (12)

Since \(\alpha_i^k - \alpha_i^{k-1} - \cdots - 1 = 0\), we can multiply by \(\alpha_i - 1\) to get \(\alpha_i^{k+1} - 2\alpha_i^k = -1\), which implies \((\alpha_i - 2) = -1 \cdot \alpha_i^{-k}\). We use this last equation to transform (12) as follows:

\[ \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} = \frac{\alpha_i - 1}{2 + (k + 1)(-\alpha_i^{-k})} = \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)} \]

This establishes the equivalence of the two formulas (11) and (12), as desired. \(\square\)

5 Proof of Theorem 2

Let \(E_n\) be as defined in Lemma 5. We wish to show that \(|E_n| < \frac{1}{2}\) for all \(n \geq 2 - k\). We proceed by first showing that \(|E_n| < \frac{1}{2}\) for \(n = 0\), then for \(n = -1, -2, -3, \ldots, 2 - k\), then for \(n = 1\), and finally that this implies \(|E_n| < \frac{1}{2}\) for all \(n \geq 2 - k\).

To begin, we note that since our initial conditions give us that \(F_n^{(k)} = 0\) for \(n = 0, -1, -2, \ldots, 2 - k\), then we need only show \(|m^{(k)}(\alpha) \cdot \alpha^{n-1}| < 1/2\) for those values of \(n\). Starting with \(n = 0\), it’s easy to check by hand that \(m^{(k)}(\alpha) \cdot \alpha^{-1} < 1/2\) for \(k = 2\) and 3, and as for \(k \geq 4\), we have the following inequality from Lemma 3:

\[ 2 - \frac{1}{3k} < \alpha, \]

which implies

\[ \alpha^{-1} < \frac{3k}{6k - 1}. \]

Also, by Lemma 4,

\[ m^{(k)}(\alpha) < m^{(k)}(2 - 1/3k) = \frac{3k - 1}{5k - 1}, \]

so thus:

\[ m^{(k)}(\alpha) \cdot \alpha^{-1} < \frac{3k - 1}{5k - 1} \cdot \frac{3k}{6k - 1} < \frac{(3k) \cdot 1}{(5k - 1) \cdot 2} < \frac{1}{2}, \]

which completes the proof.
as desired. Thus, $0 < |m^{(k)}(\alpha) \cdot \alpha^{-1}| < 1/2$ for all $k$, as desired.

Since $\alpha^{-1} < 1$, we can conclude that for $n = -1, -2, \ldots, 2 - k$, then $|E_n| = m^{(k)}(\alpha) \cdot \alpha^{n-1} < 1/2$.

Turning our attention now to $E_1$, we note that $E_1^{(k)} = 1$ (again by definition of our initial conditions) and that

$$\frac{1}{2} = m(2) < m(\alpha) < m(2 - 1/k) = 1$$

which immediately gives us $|E_1| < 1/2$.

As for $E_n$ with $n \geq 2$, we know from Lemma 5 that

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad \text{(for } n \geq 2)$$

Suppose for some $n \geq 2$ that $|E_n| \geq 1/2$. Let $n_0$ be the smallest positive such $n$. Now, subtracting the following two equations:

$$E_{n_0+1} = E_{n_0} + E_{n_0-1} + \cdots + E_{n_0-(k-1)}$$
$$E_{n_0} = E_{n_0-1} + E_{n_0-2} + \cdots + E_{n_0-k}$$

gives us:

$$E_{n_0+1} = 2E_{n_0} - E_{n_0-k}$$

Since $|E_{n_0}| \geq |E_{n_0-k}|$ (the first, by assumption, being larger than, and the second smaller than, 1/2), we can conclude that $|E_{n_0+1}| > |E_{n_0}|$. In fact, we can apply this argument repeatedly to show that $|E_{n_0+i}| > \cdots > |E_{n_0+1}| > |E_{n_0}|$. However, this contradicts the observation from Eq. (3) that the error must eventually go to 0. We conclude that $|E_n| < 1/2$ for all $n \geq 2$, and thus for all $n \geq 2 - k$. \hfill \Box

6 Acknowledgement

The first author would like to thank J. Siehler for inspiring this paper with his work on Tribonacci numbers.

References


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*2000 Mathematics Subject Classification: Primary 11B39, Secondary 11C08, 33F05, 65D20. Keywords: $k$-generalized Fibonacci numbers, Binet form, Tribonacci number, Tetranacci number, Pentanacci number.*

(Concerned with sequences *A000073, A000078,* and *A001591.*

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Received October 15 2008; revised versions received February 19 2014; February 23 2014. Published in *Journal of Integer Sequences,* February 26 2014.

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