Fermat Numbers in Multinomial Coefficients

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Abstract
In 2001 Luca proved that no Fermat number can be a nontrivial binomial coefficient. We extend this result to multinomial coefficients.

1 Introduction
Let \( F_m = 2^{2^m} + 1 \) be the \( m \)th Fermat number for any nonnegative integer \( m \). Several authors studied the Diophantine equation

\[
\binom{n}{k} = 2^{2^m} + 1 = F_m,
\]

where \( n \geq 2k \geq 2, \) and \( m \geq 0 \). We refer to the articles [2, 3, 5, 6, 8] for further details. In 2001, Luca [6] completely solved Eq. (1) and proved that it has only the trivial solutions \( k = 1, n - 1 \) and \( n = F_m \). The proof is mainly based on a congruence given by Lucas [7]. For more about Fermat numbers, see [4].

For a positive integer \( t \), let \( n, k_1, \ldots, k_t \) be nonnegative integers, and define the \( t \)-order multinomial coefficient as follows:

\[
\binom{n}{k_1, \ldots, k_t} = \frac{n(n-1) \cdots (n-k_1-\cdots-k_t+1)}{k_1! \cdots k_t!},
\]

with \( \sum_{i=1}^t k_i < n + 1 \). In particular, \( \binom{n}{0, \ldots, 0} = 1 \). Note that for \( t \geq 2 \), if \( \sum_{i=1}^t k_i = n \), then the \( t \)-order multinomial coefficient equals a \( (t-1) \)-order multinomial coefficient

\[
\binom{n}{k_1, \ldots, k_t} = \binom{n}{k_1, \ldots, k_{t-1}}.
\]
There are many papers concerning the Diophantine equations related to multinomial coefficients. For example, Yang and Cai [9] proved that the Diophantine equation
\[
\binom{n}{k_1, \ldots, k_t} = x^l
\]
has no positive integer solutions for \(n, t \geq 3, l \geq 2,\) and \(\sum_{i=1}^{t} k_i = n.\)

In this paper, we consider the Diophantine equation
\[
\binom{n}{k_1, \ldots, k_t} = 2^{2m} + 1 = F_m, \quad \text{for } t \geq 2, \text{ and } \sum_{i=1}^{t} k_i < n,
\]
and prove the following theorem.

**Theorem 1.** The Diophantine equation (2) has no integer solutions \((m, n, k_1, \ldots, k_t)\) for nonnegative \(m\) and positive \(n, k_1, \ldots, k_t.\)

## 2 Two Lemmas

To prove Theorem 1, we need the following two lemmas.

**Lemma 2** (Euler [1]). Any prime factor \(p\) of the Fermat number \(F_m\) satisfies
\[
p \equiv 1 \pmod{2^{m+1}}.
\]

**Lemma 3** (Luca [6]). If \(F_m = s\binom{n}{k},\) with \(m \geq 5, s \geq 1,\) and \(1 \leq k \leq \frac{n}{2},\) we have the following two properties.

(i) Let \(n = n'd,\) where
\[
A = \{p : \text{prime } p \mid n, \text{ and } p \equiv 1 \pmod{2^{m+1}}\},
\]
and
\[
n' = \prod_{p \in A} p^{\alpha_p}.
\]
Then \(k = d < 2^m.\)

(ii) \(k - i \mid n - i\) for any \(i = 0, \ldots, k - 1.\)

**Remark 4.** Lemma 3 is summarized from Luca’s proof [6] of Diophantine equation (1). Although Luca only proved the case \(s = 1,\) he indicated that the result also holds for all positive integers \(s.\)
3 Proof of Theorem 1

The first five Fermat numbers are primes, which cannot be a multinomial coefficient in Eq. (2). Therefore, we only need to consider $m \geq 5$.

Moreover, for any multinomial coefficient $\binom{n}{k_1, \ldots, k_t}$ with $t > 0$, $k_1, \ldots, k_t \geq 1$, and $\sum_{i=1}^t k_i < n$, there exists a multinomial coefficient

$$\binom{n}{k'_1, \ldots, k'_t} = \binom{n}{k_1, \ldots, k_t},$$

such that $1 \leq k'_1, \ldots, k'_t \leq \frac{n}{2}$.

Hence, Eq. (2) becomes

$$F_m = \binom{n}{k_1, \ldots, k_t}, \quad \text{for } m \geq 5, \ 1 \leq k_i \leq \frac{n}{2}, \ \text{and } \sum_{i=1}^t k_i < n. \quad (3)$$

Let $n = n'd$, where

$$A = \{ p : \text{prime } p \mid n, \ \text{and } p \equiv 1 \pmod{2^{m+1}} \},$$

and

$$n' = \prod_{p \in A} p^{\alpha_p}.$$

For any $i = 1, \ldots, t$, we have

$$F_m = \binom{n}{k_1, \ldots, k_t} = \binom{n}{k_i} \binom{n-k_i}{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_t}, \quad (4)$$

where $\binom{n-k_i}{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_t}$ is a positive integer. By Lemma 3 (i), we have $k_i = d < 2^m$ for $i = 1, \ldots, t$. Then Eq. (3) becomes

$$F_m = \binom{n}{d, \ldots, d}, \quad n > td, \ \text{and } t \geq 2$$

$$= \binom{n}{d} \binom{n-d}{d} \binom{n-2d}{d, \ldots, d}. \quad (5)$$

Note that $d \geq 1$. We study Eq. (6) in the following three cases.

Case 1: $d > 2$. Since $n > 2d$ and $d \mid n$, we have $n \geq 3d$. Then,

$$d \leq \frac{n-d}{2} < \frac{n}{2}.$$
In Eq. (6), applying Lemma 3 (ii) to \( \binom{n}{d} \) and \( \binom{n-d}{d} \) respectively, and setting \( i = 1 \), we have

\[ d - 1 \mid n - 1 \]

and

\[ d - 1 \mid n - d - 1. \]

Thus, \( d - 1 \mid d \), which is impossible.

Case 2: \( d = 2 \). Let \( n = 2n' \). Then Eq. (5) becomes

\[
F_m = \left( \frac{2n'}{2, \ldots, 2}_t \right) = n'(2n' - 1)(n' - 1)(2n' - 3) \binom{2n' - 4}{2, \ldots, 2}_{t-2}.
\]

Then \( n' \) and \( n' - 1 \) are both \( F_m \)'s factors. According to Lemma 2, we obtain \( n' \equiv n' - 1 \equiv 1 \pmod{2^{m+1}} \), which is impossible.

Case 3: \( d = 1 \). Eq. (5) becomes

\[
F_m = \binom{n}{1, \ldots, 1}_t = n(n - 1) \binom{n - 2}{1, \ldots, 1}_{t-2}.
\]

Then \( n \) and \( n - 1 \) are both \( F_m \)'s factors. According to Lemma 2, we obtain \( n \equiv n - 1 \equiv 1 \pmod{2^{m+1}} \), which is also impossible.

This completes the proof of Theorem 1.

Remark 5. One can even find that the multinomial coefficient in Eq. (2) could not divide a Fermat number. Otherwise, assume that there exists a positive integer \( s \) such that

\[
F_m = s\binom{n}{k_1, \ldots, k_t}.
\]

Note that in Eq. (4) we still have

\[
\binom{n}{k_i} \mid F_m,
\]

and in Eqs. (5) and (6) similar results hold. Hence, we can get the proof in the same way.

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References


[9] P. Yang and T. Cai, On the Diophantine equation \( \left( \binom{n}{k_1,\ldots,k_s} \right) = x^l \), *Acta Arith.* 151 (2012), 7–9.

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