Constructing Exponential Riordan Arrays from Their $A$ and $Z$ Sequences

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Abstract

We show how to construct an exponential Riordan array from a knowledge of its $A$ and $Z$ sequences. The effect of pre- and post-multiplication by the binomial matrix on the $A$ and $Z$ sequences is examined, as well as the effect of scaling the $A$ and $Z$ sequences. Examples are given, including a discussion of related Sheffer orthogonal polynomials.

1 Introduction

One of the most fundamental results concerning Riordan arrays is that they have a sequence characterization [13, 18]. This normally involves two sequences, called the $A$-sequence and the $Z$-sequence. For exponential Riordan arrays [9] (see Appendix), this characterization is equivalent to the fact that the production matrix [11] of an exponential array $[g, f]$, with $A$-sequence $A(t)$ and $Z$-sequence $Z(t)$ has bivariate generating function

$$e^{zt}(Z(t) + A(t)z).$$

In this case we have

$$A(t) = f'(\tilde{f}(t)), \quad Z(t) = \frac{g'(\tilde{f}(t))}{g(\tilde{f}(t))}.$$  

Examples of exponential Riordan arrays and their production matrices may be found in the On-Line Encyclopedia of Integer Sequences [19, 20]. In that database, sequences are referred to by their $A$-numbers. For known sequences, we shall adopt this convention in this note.
A natural question to ask is the following. If we are given two suitable power series \( A(t) \) and \( Z(t) \), can we recover the corresponding exponential Riordan array \([g(t), f(t)]\) whose \( A \) and \( Z \) sequences correspond to the given power series \( A(t) \) and \( Z(t) \)?

The next two simple results provide a means of doing this.

**Lemma 1.** For an exponential Riordan array \([g(t), f(t)]\) with \( A \)-sequence \( A(t) \), we have

\[
\frac{d}{dt} \bar{f}(t) = \frac{1}{A(t)}.
\]

*Proof.* By definition of the compositional inverse, we have

\[
f(\bar{f}(t)) = t.
\]

Differentiating this with respect to \( t \), we obtain

\[
f'(\bar{f}(t)) \frac{d}{dt} \bar{f}(t) = 1
\]

or

\[
\frac{d}{dt} \bar{f}(t) = \frac{1}{f'(\bar{f}(t))} = \frac{1}{A(t)}.
\]

\(\square\)

**Lemma 2.** For an exponential Riordan array \([g(t), f(t)]\) with \( A \)-sequence \( A(t) \) and \( Z \)-sequence \( Z(t) \), we have

\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)}.
\]

*Proof.* We have

\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))} \frac{d}{dt} \bar{f}(t) = Z(t) \frac{1}{A(t)} = \frac{Z(t)}{A(t)}.
\]

\(\square\)

Thus if we can easily carry out the reversion from \( \bar{f}(t) \) to \( f(t) \), a knowledge of \( A(t) \) and \( Z(t) \), along with the equations

\[
\frac{d}{dt} \bar{f}(t) = \frac{1}{A(t)}, \quad \frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)}
\]

will allow us to find \( f(t) \) and \( g(t) \). The steps to achieve this are as follows.

- Using the equation \( \frac{d}{dt} \bar{f}(t) = \frac{1}{A(t)} \), solve for \( \bar{f}(t) \).
- Revert \( \bar{f}(t) \) to get \( f(t) \).
Solve the equation \( \frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} \) and take the exponential to get \( g(\bar{f}(t)) \).

Solve for \( g(t) \) by substituting \( f(t) \) in place of \( t \) in the last found expression.

Constants of integration may be determined using such conditions as \( \bar{f}(0) = f(0) = 0 \), and \( g(0) = 1 \).

**Example 3.** We seek to find \([g(t), f(t)]\) where

\[
A(t) = \frac{1}{1 + t}, \quad Z(t) = -\frac{1}{1 + t}.
\]

We start by solving the equation

\[
\frac{d}{dt} \bar{f}(t) = 1 + t.
\]

Since \( \bar{f}(0) = 0 \), we find that

\[
\bar{f}(t) = t + \frac{t^2}{2} = t \left(1 + \frac{t}{2}\right).
\]

We revert this to get

\[
f(t) = \sqrt{1 + 2t} - 1.
\]

We now solve the equation

\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = -1.
\]

Thus we find that

\[
\ln(g(\bar{f}(t))) = -t \Rightarrow g(\bar{f}(t)) = e^{-t}.
\]

Thus (since \( \bar{f}(f(t)) = t \)) we get

\[
g(t) = e^{-f(t)} = e^{1 - \sqrt{1 + 2t}}.
\]

Hence the exponential Riordan array with the given \( A \) and \( Z \) sequences is

\[
[g, f] = \left[e^{1 - \sqrt{1 + 2t}}, \sqrt{1 + 2t} - 1\right].
\]

We note that

\[
[g, f]^{-1} = \left[e^t, t + \frac{t^2}{2}\right]
\]

which is the Pascal-like matrix \([6].\)

In like manner, we can show that

\[
A(t) = \frac{1}{1 + 2t}, \quad Z(t) = -\frac{1}{1 + 2t}
\]
corresponds to the exponential Riordan array
\[ \left[ g, f \right] = \left[ e^{\frac{1-\sqrt{1+4t}}{2}}, \frac{\sqrt{1+4t} - 1}{2} \right], \]
whose inverse
\[ \left[ g, f \right]^{-1} = \left[ e^t, t + t^2 \right] \]
is Pascal-like [6]. In general, if \( A(t) = -Z(t) = \frac{1}{1+rt} \), then
\[ \left[ g, f \right] = \left[ e^{\frac{1}{r}(1-\sqrt{1+2rt})}, \frac{1}{r}(\sqrt{1+2rt} - 1) \right]. \]
Then
\[ \left[ g, f \right]^{-1} = \left[ e^t, t + \frac{t^2}{2} \right] \]
is a Pascal-type matrix.

2 Effect of the binomial transform

The next proposition shows the effect of changing \( Z(t) \) to \( Z(t) + 1 \) and to \( Z(t) + A(t) \), respectively. We recall that the binomial matrix \( B = [e^t, t] \).

**Proposition 4.** Let \( [g, f] \) be an exponential Riordan array with \( A \) and \( Z \) sequences \( A(t) \) and \( Z(t) \) respectively. Then the exponential Riordan array \( B \cdot [g, f] \) has \( A \) and \( Z \) sequences \( A(t) \) and \( Z(t) + 1 \) respectively, while the exponential Riordan array \( [g, f] \cdot B \) has \( A \) and \( Z \) sequences \( A(t) \) and \( Z(t) + A(t) \) respectively.

**Proof.** Firstly, we let the exponential Riordan array \( [h, l] \) have \( A \) and \( Z \) sequences \( A(t) \) and \( Z(t) + 1 \) respectively. Then we have \( \frac{d}{dt} \bar{l}(t) = \frac{1}{A(t)} \), which implies that \( l(t) = f(t) \) (since \( l(0) = f(0) = 0 \)). Now
\[ \frac{d}{dt} \ln(h(\bar{l}(t))) = \frac{d}{dt} \ln(h(\bar{f}(t))) = \frac{Z(t) + 1}{A(t)} = \frac{Z(t)}{A(t)} + \frac{1}{A(t)}. \]
Thus
\[ \ln(h(\bar{f}(t))) = \ln(g(\bar{f}(t))) + \bar{f}(t) \Rightarrow h(\bar{f}(t)) = g(\bar{f}(t))e^{\bar{f}(t)}. \]
We obtain that
\[ h(t) = g(t)l^t \]
and so
\[ [h(t), l(t)] = [e^t g(t), f(t)] = [e^t, t] \cdot [g(t), f(t)] = B \cdot [g(t), f(t)]. \]
Secondly, we now assume that the exponential Riordan array \([h, l] \) have A and Z sequences \(A(t)\) and \(Z(t) + A(t)\) respectively. As before, we see that \(l(t) = f(t)\). Also,

\[
\frac{d}{dt} \ln(h(l(t))) = \frac{d}{dt} \ln(h(\bar{f}(t))) = \frac{Z(t) + A(t)}{A(t)} = \frac{Z(t)}{A(t)} + 1.
\]

Thus

\[
\ln(h(\bar{f}(t))) = \ln(g(\bar{f}(t))) + t \Rightarrow h(\bar{f}(t)) = g(\bar{f}(t))e^t.
\]

Now substituting \(f(t)\) for \(t\) gives us

\[
h(t) = e^{f(t)}g(t).
\]

Thus

\[
[h, l] = [e^{f(t)}g(t), f(t)] = [g(t), f(t)] \cdot [e^t, t] = [g(t), f(t)] \cdot B.
\]

We shall see examples of these results in the next section.

\[\square\]

### 3 Effect of Scaling

In this section, we will assume that the exponential Riordan array with A and Z sequences \(A(t)\) and \(Z(t)\), respectively, is given by \([g(t), f(t)]\). We wish to characterize the exponential Riordan array \([g^*(t), f^*(t)]\) whose A and Z sequences are \(A^*(t) = rA(t)\) and \(Z^*(t) = sZ(t)\) respectively.

**Proposition 5.** We have

\[
[g^*(t), f^*(t)] = [g(rt)^\frac{1}{r}, rf(t)].
\]

**Proof.** We have

\[
\frac{d}{dt} f^*(t) = \frac{1}{rA} = \frac{1}{r} \frac{d}{dt} \bar{f}(t).
\]

Thus

\[
\bar{f}^*(t) = \frac{1}{r} \bar{f}(t) \Rightarrow f^*(t) = rf(t).
\]

Then

\[
\frac{d}{dt} \ln(g^*(\bar{f}^*(t))) = \frac{sZ}{rA} = \frac{s}{r} \frac{d}{dt} \ln(g(\bar{f}(t))),
\]

and so

\[
\ln(g^*(\bar{f}^*(t))) = \frac{s}{r} \ln(g(\bar{f}(t))) = \ln \left( g(\bar{f}(t))^\frac{s}{r} \right).
\]

Thus

\[
g^*(\bar{f}^*(t)) = g(\bar{f}(t))^\frac{s}{r} \Rightarrow g^*(\frac{1}{r} \bar{f}(t)) = g(\bar{f}(t))^\frac{s}{r} \Rightarrow g^*(\frac{1}{r} t) = g(t)^\frac{s}{r},
\]

or

\[
g^*(t) = g(rt)^\frac{s}{r}.
\]
Example 6. We let 
\[ A(t) = 1 + t, \quad Z(t) = 1 + 2t. \]
We find that the corresponding exponential array is
\[ [g, f] = \left[ e^{2e^t - t - 2}, e^t - 1 \right], \]
which begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
3 & 3 & 1 & 0 & 0 & 0 & \cdots \\
9 & 13 & 6 & 1 & 0 & 0 & \cdots \\
35 & 59 & 37 & 10 & 1 & 0 & \cdots \\
153 & 301 & 230 & 85 & 15 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]
with production matrix which begins
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 2 & 1 & 0 & 0 & 0 & \cdots \\
0 & 4 & 3 & 1 & 0 & 0 & \cdots \\
0 & 0 & 6 & 4 & 1 & 0 & \cdots \\
0 & 0 & 0 & 8 & 5 & 1 & \cdots \\
0 & 0 & 0 & 0 & 10 & 6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}.
\]
We now take 
\[ A^*(t) = 3(1 + t), \quad Z^*(t) = 5(1 + 2t). \]
The corresponding exponential Riordan array is then given by
\[ [g^*(t), f^*(t)] = \left[ \left( e^{2e^{3t - 3t - 2}} \right)^{\frac{3}{2}}, 3(e^t - 1) \right]. \]
This array begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
5 & 3 & 0 & 0 & 0 & 0 & \cdots \\
55 & 33 & 9 & 0 & 0 & 0 & \cdots \\
665 & 543 & 162 & 27 & 0 & 0 & \cdots \\
9895 & 9033 & 3573 & 702 & 81 & 0 & \cdots \\
165185 & 170103 & 76410 & 19575 & 2835 & 243 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]
with production matrix which begins

\[
\begin{pmatrix}
5 & 3 & 0 & 0 & 0 & 0 & \ldots \\
10 & 8 & 3 & 0 & 0 & 0 & \ldots \\
0 & 20 & 11 & 3 & 0 & 0 & \ldots \\
0 & 0 & 30 & 14 & 3 & 0 & \ldots \\
0 & 0 & 0 & 40 & 17 & 3 & \ldots \\
0 & 0 & 0 & 0 & 50 & 20 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

4 Further examples

**Example 7.** We take the Stirling number related choice of

\[ A(t) = 1 + t, \quad Z(t) = 1 + t. \]

From

\[ \frac{d}{dt} \bar{f}(t) = \frac{1}{1 + t}, \]

we obtain

\[ \bar{f}(t) = \ln(1 + t) \Rightarrow f(t) = e^t - 1. \]

Then from

\[ \frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = 1 \]

we obtain

\[ \ln(g(\bar{f}(t))) = t \Rightarrow g(\bar{f}(t)) = e^t, \]

and hence

\[ g(t) = e^{e^t - 1}. \]

Thus we obtain

\[ [g, f] = [e^{e^t - 1}, e^t - 1], \]

which is A049020. We have

\[ [g, f] = S_2 \cdot B \]
where \( S_2 \) is the matrix of Stirling numbers of the second kind (A048993) and \( B \) is the binomial matrix (A007318). The production array of \([g, f]\) is given by

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 3 & 1 & 0 & 0 & \ldots \\
0 & 0 & 3 & 4 & 1 & 0 & \ldots \\
0 & 0 & 0 & 4 & 5 & 1 & \ldots \\
0 & 0 & 0 & 0 & 5 & 6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Since this production matrix is tri-diagonal, the inverse matrix \([g, f]^{-1}\) is the coefficient array of a family of orthogonal polynomials [4, 3]. The family in question is the family of Charlier polynomials, which has the Bell numbers (with e.g.f. \( e^{e^t-1} \)) as moments. The Charlier polynomials satisfy the three-term recurrence

\[
P_n(t) = (t - n)P_{n-1}(t) - (n - 1)P_{n-2}(t),
\]

with \( P_0(t) = 1, \ P_1(t) = t - 1. \)

**Example 8.** We take

\[
A(t) = 1 + t \quad Z(t) = 1 + t + t^2.
\]

Again, we find that

\[
f(t) = e^t - 1.
\]

Then

\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1 + t + t^2}{1 + t},
\]

and hence

\[
\ln(g(\bar{f}(t))) = \frac{t^2}{2} + \ln(1 + t).
\]

Thus

\[
g(\bar{f}(t)) = e^{\frac{t^2}{2}}(1 + t),
\]

and so

\[
g(t) = e^{\frac{(e^t - 1)^2}{2}}(1 + e^t - 1) = e^t e^{\frac{(e^t - 1)^2}{2}}.
\]

In this case, the production matrix is four-diagonal and begins

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 2 & 1 & 0 & 0 & 0 & \ldots \\
2 & 2 & 3 & 1 & 0 & 0 & \ldots \\
0 & 6 & 3 & 4 & 1 & 0 & \ldots \\
0 & 0 & 12 & 4 & 5 & 1 & \ldots \\
0 & 0 & 0 & 20 & 5 & 6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
The exponential Riordan array

\[ [g, f] = \left[ e^{t \frac{(e^t - 1)^2}{2}}, e^t - 1 \right] \]

begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3 & 1 & 0 & 0 & 0 & \cdots \\
7 & 10 & 6 & 1 & 0 & 0 & \cdots \\
29 & 45 & 31 & 10 & 1 & 0 & \cdots \\
136 & 241 & 180 & 75 & 15 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The row sums of this array are the Dowling numbers A007405.

We note that the exponential Riordan array

\[ B^{-1} \cdot [g, f] = [e^{-t}, t] \cdot [g, f] = \left[ e^{t \frac{(e^t - 1)^2}{2}}, e^t - 1 \right] \]

has

\[ A(t) = 1 + t \quad Z(t) = t + t^2. \]

This array begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
3 & 4 & 3 & 1 & 0 & 0 & \cdots \\
10 & 19 & 13 & 6 & 1 & 0 & \cdots \\
45 & 91 & 75 & 35 & 10 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The first column of this array is A060311, while its row sums are given by A004211. The production matrix of this array begins

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
2 & 2 & 2 & 1 & 0 & 0 & \cdots \\
0 & 6 & 3 & 3 & 1 & 0 & \cdots \\
0 & 0 & 12 & 4 & 4 & 1 & \cdots \\
0 & 0 & 0 & 20 & 5 & 5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where we see that the effect of the inverse binomial matrix is to subtract 1 from the diagonal.
In this example, we have $Z(t) = 1 + t + t^2 = A(t) + t^2$. Thus the exponential Riordan array $[g, f]$ is equal to the product $[h, l] \cdot B$ where the exponential Riordan array $[h, l]$ has $A$ and $Z$ sequences of $1+t$ and $t^2$, respectively.

**Example 9.** We take

$$A(t) = 1 + t^2, \quad Z(t) = 1 + t + t^2.$$  

Then Thus

$$f(t) = \tan(t).$$  

Now

$$\frac{d}{dt} \ln(g(\tilde{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1 + t + t^2}{1 + t^2} = \frac{1}{1 + t^2},$$

and so

$$\ln(g(\tilde{f}(t))) = \ln\sqrt{1 + t^2} + t.$$  

Thus

$$g(\tilde{f}(t)) = e^t\sqrt{1 + t^2} \Rightarrow g(t) = e^{\tan(t)}\sqrt{1 + \tan^2(t)} = \frac{e^{\tan(t)}}{\cos(t)}.$$  

Thus the sought-for exponential Riordan array is given by

$$[g, f] = [e^{\tan(t)} \sec(t), \tan(t)].$$

This matrix begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 2 & 1 & 0 & 0 & 0 & \cdots \\
6 & 8 & 3 & 1 & 0 & 0 & \cdots \\
20 & 32 & 20 & 4 & 1 & 0 & \cdots \\
92 & 156 & 100 & 40 & 5 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

with production matrix that begins

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
2 & 4 & 1 & 1 & 0 & 0 & \cdots \\
0 & 6 & 9 & 1 & 1 & 0 & \cdots \\
0 & 0 & 12 & 16 & 1 & 1 & \cdots \\
0 & 0 & 0 & 20 & 25 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$


10
The first column is A009244. We note that we have the following factorization
\[ [g, f] = [e^{\tan(t)} \sec(t), \tan(t)] = [\sec(t), \tan(t)] \cdot B. \]

Thus we can say that the exponential Riordan array \([\sec(t), \tan(t)]\), which begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 5 & 0 & 1 & 0 & \cdots \\
5 & 0 & 14 & 0 & 1 & \cdots \\
0 & 61 & 0 & 30 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

has A sequence defined by \(1 + t^2\) and Z sequence defined by \(t\). Thus its production matrix is given by
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 4 & 0 & 1 & 0 & \cdots \\
0 & 0 & 9 & 0 & 1 & \cdots \\
0 & 0 & 0 & 16 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 25 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

We can infer from this that the inverse array
\[
[\sec(t), \tan(t)]^{-1} = \left[ \frac{1}{\sqrt{1 + t^2}}, \tan^{-1}(t) \right]
\]
is the coefficient array of the family of orthogonal polynomials
\[
P_n(t) = tP_{n-1}(t) - (n - 1)^2 P_{n-2}(t),
\]
with \(P_0(t) = 1\) and \(P_1(t) = t\).

**Example 10.** In this example, we let
\[
A(t) = 1 + t, \quad Z(t) = \frac{1}{1 - t}. 
\]

As before, we get \(f(t) = e^t - 1\). Now
\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1}{1 - t^2},
\]

11
and hence
\[ \ln(g(\bar{f}(t))) = \frac{1}{2} \ln \left( \frac{1+t}{1-t} \right). \]

We infer that
\[ g(t) = \sqrt{\frac{e^t}{2 - e^t}}. \]

The function \( g(t) \) generates the sequence \texttt{A014307} which begins
\[ 1, 1, 2, 7, 35, 226, 1787, 16717, 180560, 2211181, 30273047, \ldots. \]

It has many combinatorial interpretations [7, 15, 17].

The exponential Riordan array
\[ [g, f] = \left[ \sqrt{\frac{e^t}{2 - e^t}}, e^t - 1 \right] \]

begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3 & 1 & 0 & 0 & 0 & \cdots \\
7 & 10 & 6 & 1 & 0 & 0 & \cdots \\
35 & 45 & 31 & 10 & 1 & 0 & \cdots \\
226 & 271 & 180 & 75 & 15 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

with production matrix that begins
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & \cdots \\
2 & 2 & 3 & 1 & 0 & 0 & \cdots \\
6 & 6 & 3 & 4 & 1 & 0 & \cdots \\
24 & 24 & 12 & 4 & 5 & 1 & \cdots \\
120 & 120 & 60 & 20 & 5 & 6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

In general, the exponential Riordan array with
\[ A(t) = 1 + t, \quad Z(t) = \frac{r}{1 - t}, \]

is given by
\[ [g, f] = \left[ \left( \frac{e^t}{2 - e^t} \right)^{r/2}, e^t - 1 \right]. \]
Example 11. For this example, we take

\[ A(t) = e^{-t}, \quad Z(t) = e^t. \]

Then

\[ \frac{d}{dt} \bar{f}(t) = \frac{1}{A(t)} = \frac{1}{e^{-t}} = e^t, \]

and so we get

\[ \bar{f}(t) = e^t + C = e^t - 1 \]

since \( \bar{f}(0) = 0 \). Thus

\[ f(t) = \ln(1 + t). \]

Now

\[ \frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{e^t}{e^{-t}} = e^{2t}, \]

and so

\[ \ln(g(\bar{f}(t))) = \frac{e^{2t}}{2} - \frac{1}{2} \Rightarrow g(\bar{f}(t)) = e^{\frac{1}{2}(e^{2t} - 1)}. \]

Substituting \( f(t) \) for \( t \) we get

\[ g(t) = e^{\frac{1}{2}(e^{2\ln(1+t)} - 1)} = e^{t + \frac{t^2}{2}}. \]

Thus

\[ [g, f] = \left[ e^{t + \frac{t^2}{2}}, \ln(1 + t) \right]. \]

We note that if we have

\[ A(t) = Z(t) = e^{-t}, \]

then we obtain

\[ [g, f] = [1 + t, \ln(1 + t)]. \]

Interestingly, this last exponential Riordan array has a production matrix that is equal the ordinary Riordan array

\[
\begin{pmatrix}
\frac{1+2t}{1+t} & t \\
\frac{t}{1+t} & 
\end{pmatrix}
\]

with its first row removed.
5 Orthogonal polynomials

When \( Z(t) = \alpha + \beta t \) and \( A(t) = 1 + \gamma t + \delta t^2 \), the production matrix of the corresponding exponential Riordan array \([g, f]\) is tri-diagonal, beginning as follows.

\[
\begin{pmatrix}
\alpha & 1 & 0 & 0 & 0 & 0 & \ldots \\
\beta & \alpha + \gamma & 1 & 0 & 0 & 0 & \ldots \\
0 & 2(\beta + \delta) & \alpha + 2\gamma & 1 & 0 & 0 & \ldots \\
0 & 0 & 3(\beta + 2\delta) & \alpha + 3\gamma & 1 & 0 & \ldots \\
0 & 0 & 0 & 4(\beta + 3\delta) & \alpha + 4\gamma & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

As a consequence, \([g, f]^{-1}\) is the coefficient array of the family of orthogonal polynomials \( P_n(t) \) defined by the three-term recurrence \([8, 12, 21]\)

\[P_n(t) = (t - (\alpha + (n - 1)\gamma))P_{n-1}(t) - (n - 1)(\beta + (n - 2)\delta)P_{n-2}(t),\]

with \( P_0(t) = 1 \) and \( P_1(t) = x - \alpha \). These are precisely the Sheffer orthogonal polynomials \([1, 13]\).

Example 12. We take the case of

\[A(t) = 1 + t + t^2, \quad Z(t) = 1 + t.\]

We have

\[
\frac{d}{dt}\tilde{f}(t) = \frac{1}{1 + t + t^2}.
\]

Choosing the constant of integration so that \( \tilde{f}(0) = 0 \), we get

\[
\tilde{f}(t) = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2t + 1}{\sqrt{3}} \right) - \frac{\pi}{3\sqrt{3}}.
\]

Thus

\[
f(t) = \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}t}{2} + \frac{\pi}{6} \right) - \frac{1}{2}
\]

\[
= \frac{2 \sin \left( \frac{\sqrt{3}t}{2} \right)}{\sqrt{3} \cos \left( \frac{\sqrt{3}t}{2} \right) - \sin \left( \frac{\sqrt{3}t}{2} \right)}
\]

\[
= \frac{2 \tan \left( \frac{\sqrt{3}t}{2} \right)}{\sqrt{3} - \tan \left( \frac{\sqrt{3}t}{2} \right)}.
\]
We now have
\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1 + t}{1 + t + t^2},
\]
and hence
\[
\ln(g(\bar{f}(t))) = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + \frac{1}{2} \ln(1 + t + t^2) - \frac{\pi}{6\sqrt{3}}.
\]
From this we infer that
\[
g(t) = \frac{\sqrt{3}e^{\frac{\sqrt{3}t}{2}}}{\sqrt{3} \cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}.
\]
The function \(g(t)\) generates the sequence \(A049774\), which counts the number of permutations of \(n\) elements not containing the consecutive pattern 123.

The sought-for matrix is thus
\[
[g, f] = \begin{bmatrix}
\frac{\sqrt{3}e^{\frac{\sqrt{3}t}{2}}}{\sqrt{3} \cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)}, & 2\sin\left(\frac{\sqrt{3}t}{2}\right) \\
\sqrt{3} \cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right), & \sqrt{3} \cos\left(\frac{\sqrt{3}t}{2}\right) - \sin\left(\frac{\sqrt{3}t}{2}\right)
\end{bmatrix}.
\]
This exponential Riordan array is \(A182822\), which begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3 & 1 & 0 & 0 & 0 & \cdots \\
5 & 12 & 6 & 1 & 0 & 0 & \cdots \\
17 & 53 & 39 & 10 & 1 & 0 & \cdots \\
70 & 279 & 260 & 95 & 15 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
with production matrix that begins
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & 0 & \cdots \\
0 & 4 & 3 & 1 & 0 & 0 & \cdots \\
0 & 0 & 9 & 4 & 1 & 0 & \cdots \\
0 & 0 & 0 & 16 & 5 & 1 & \cdots \\
0 & 0 & 0 & 0 & 25 & 6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

**Example 13.** We change the previous example slightly by taking
\[
A(t) = 1 + 2t + t^2 = (1 + t)^2, \quad Z(t) = 1 + t.
\]
Then we have
\[
\frac{d}{dt} \bar{f}(t) = \frac{1}{(1+t)^2} \Rightarrow \bar{f}(t) = -\frac{1}{1+t} + 1 = \frac{t}{1+t}.
\]
This means that
\[
f(t) = \frac{t}{1-t}.
\]
Now we have
\[
\frac{d}{dt} \ln(g(\bar{f}(t))) = \frac{Z(t)}{A(t)} = \frac{1}{1+t},
\]
and hence
\[
\ln(g(\bar{f}(t))) = \ln(1+t) \Rightarrow g(\bar{f}(t)) = 1+t.
\]
This implies that
\[
g(t) = 1 + f(t) = 1 + \frac{t}{1-t} = \frac{1}{1-t}.
\]
Thus
\[
[g, f] = \left[ \frac{1}{1-t}, \frac{t}{1-t} \right]
\]
Thus \([g, f]^{-1}\) is the coefficient array of the Laguerre polynomials [5].

We finish by noting that the simple addition of \(t\) to \(A(t)\) has allowed us to go from the relatively complicated exponential Riordan array
\[
\begin{bmatrix}
\sqrt{3} e^{\frac{x^2}{2}} \\
\sqrt{3} \cos \left( \frac{\sqrt{3} t}{2} \right) - \sin \left( \frac{\sqrt{3} t}{2} \right) \\
\sqrt{3} \cos \left( \frac{\sqrt{3} t}{2} \right) - \sin \left( \frac{\sqrt{3} t}{2} \right)
\end{bmatrix}
\]
to the simple exponential Riordan array
\[
\begin{bmatrix}
\frac{1}{1-t}, t \\
\frac{1}{1-t}
\end{bmatrix}
\]

6 Appendix: exponential Riordan arrays

The exponential Riordan group [6, 9, 11], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions \(g(t) = g_0 + g_1 t + g_2 t^2 + \cdots\) and \(f(t) = f_1 t + f_2 t^2 + \cdots\) where \(g_0 \neq 0\) and \(f_1 \neq 0\). We usually assume that \(g_0 = f_1 = 1\).

The associated matrix is the matrix whose \(i\)-th column has exponential generating function \(g(t)f(t)^i/i!\) (the first column being indexed by 0). The matrix corresponding to the pair \(f, g\) is denoted by \([g, f]\). The group law is given by
\[
[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].
\]
The identity for this law is \( I = [1, t] \) and the inverse of \( [g, f] \) is \( [g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}] \) where \( \bar{f} \) is the compositional inverse of \( f \).

If \( M \) is the matrix \([g, f]\), and \( u = (u_n)_{n \geq 0} \) is an integer sequence with exponential generating function \( U(t) \), then the sequence \( Mu \) has exponential generating function \( g(t)U(f(t)) \). Thus the row sums of the array \([g, f]\) have exponential generating function given by \( g(t)e^{f(t)} \) since the sequence \( 1, 1, 1, \ldots \) has exponential generating function \( e^t \).

As an element of the group of exponential Riordan arrays, the binomial matrix \( B \) with \((n, k)\)-th element \( \binom{n}{k} \) is given by \( B = [e^t, t] \). By the above, the exponential generating function of its row sums is given by \( e^te^t = e^{2t} \), as expected (\( e^{2t} \) is the e.g.f. of \( 2^n \)).

To each exponential Riordan array \( L = [g, f] \) is associated \([10, 11]\) a matrix \( P \) called its production matrix, which has bivariate g.f. given by

\[
e^{zt}(Z(t) + A(t)z)
\]

where

\[
A(t) = f'(\bar{f}(t)), \quad Z(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}.
\]

We have

\[
P = L^{-1} \bar{L}
\]

where \( L \) \([16, 22]\) is the matrix \( L \) with its top row removed.

The ordinary Riordan group is described in \([18]\).

References


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