On the Diophantine Equation

\[ x^4 + y^4 + z^4 + t^4 = w^2 \]

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Abstract
To our knowledge, only three parametric solutions to the equation \( x^4 + y^4 + z^4 + t^4 = w^2 \) were previously known. In this paper, we study the equation \( x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \). We prove that it is possible to obtain infinitely many parametric solutions by finding points on an elliptic curve over a field \( \mathbb{Q}(m) \) and we give several new parametric solutions.

1 Introduction

Jacobi and Madden [3] considered the equation

\[ x^4 + y^4 + z^4 + t^4 = (x + y + z + t)^4. \tag{1} \]
They showed the existence of infinitely many integral solutions to (1). This is a special case of the equation
\[ x^4 + y^4 + z^4 + t^4 = w^4, \tag{2} \]
for which Elkies [1] found an infinite family of integral solutions when \( t = 0 \). In this paper, we consider a special case of a similar equation
\[ x^4 + y^4 + z^4 + t^4 = w^2. \tag{3} \]

2 Background

Consider the equation (3). We say that a solution is trivial if at least three of the numbers \( x, y, z, t, w \) are zero, for instance \((x, y, z, t, w) = (x, 0, 0, 0, x^2)\). If two and only two of the numbers \( x, y, z, t \) are zero, the equation has no nontrivial solution since Fermat proved that the equation \( x^4 + y^4 = w^2 \) has no solution in nonzero integers.

The first known parametric solution is nontrivial but very elementary:
\[ (x, y, z, t, w) = (a^2, ab, b^2, ab, a^4 + b^4). \]

In the next solution, found by Fauquembergue [2], one of the numbers \( x, y, z, t, w \) is zero, for instance \( z = 0 \):
\[ (x, y, z, t, w) = (ac, bc, 0, ab, a^4 + a^2b^2 + b^4) \]
where \( a^2 + b^2 = c^2 \). The following solution was also found by Fauquembergue [2], again assuming \( a^2 + b^2 = c^2 \):
\[ (x, y, z, t, w) = (2a^2bc^3, 2ab^2c^3, (a^2 - b^2)c^4, 2ab(a^4 + b^4), \]
\[ (a^6 + 2a^5b + 3a^4b^2 + 3a^2b^4 + 2ab^5 + b^6)(a^6 - 2a^5b + 3a^4b^2 + 3a^2b^4 - 2ab^5 + b^6)). \]

These three parametric solutions yield nontrivial numerical solutions, unless \( ab = 0 \).

3 The Equation \( x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \)

While investigating solutions to equation (3), the second author noticed some interesting properties. After some numerical results, he considered the following three cases.

- If \( w = x^2 + y^2 + z^2 + t^2 \) then \( x^2y^2 + x^2z^2 + z^2t^2 + y^2z^2 + y^2t^2 + z^2t^2 = 0 \); hence, there are only trivial solutions.
- If \( w = x^2 + y^2 - z^2 - t^2 \) then \( x^2y^2 - x^2z^2 - y^2z^2 = t^2(x^2 + y^2 - z^2) \). This is an interesting but complicated case. We leave this for future work.
If $w = x^2 + y^2 + z^2 - t^2$ then $x^2y^2 + x^2z^2 + y^2z^2 = t^2(x^2 + y^2 + z^2)$. This case also looked interesting and will be discussed below, beginning with the following proposition. We believe our analysis of this case is new.

Proposition 1. If $x^2 + y^2 + z^2 \neq 0$, then $(x, y, z, t)$ satisfies
\[ x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \] (4)
if and only if
\[ (x^2 + y^2 + z^2)(y^2z^2 + z^2x^2 + x^2y^2) = \square \] (5)
and
\[ t^2 = \frac{y^2z^2 + z^2x^2 + x^2y^2}{x^2 + y^2 + z^2}. \] (6)

Proof. We have
\[ x^4 + y^4 + z^4 + t^4 - (x^2 + y^2 + z^2 - t^2)^2 = 2((x^2 + y^2 + z^2)t^2 - (y^2z^2 + z^2x^2 + x^2y^2)). \]

Since (4) is homogeneous of degree four, from now on we will write solutions to this equation as $(x : y : z : t)$. The equation represents a surface in $\mathbb{P}^3$,
\[ \mathcal{S}' = \{(x : y : z : t) \in \mathbb{P}^3 \mid x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2\}. \]
Assuming $t \neq 0$, we can view the surface in affine form by corresponding $(x, y, z) \leftrightarrow (x : y : z : 1)$. This gives a rather interesting looking surface in three dimensions; see Figure 1.

The main focus of this paper is to answer the following question.

Question 2. How many rational curves $m \mapsto (x(m), y(m), z(m))$ are on the surface $\mathcal{S}'$?

If $xyz \neq 0$, then (5) can be expressed as
\[ (x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \square. \]

This leads to the following lemma.

Lemma 3. If $(x : y : z : t)$ is a solution to (4) such that $xyz \neq 0$ then \( \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right) \) is also a solution to (4).
Figure 1: Plot of $S'$ in affine space

**Proof.** Let $(X : Y : Z : T) = (\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t})$. Then

$$X^2 + Y^2 + Z^2 = \frac{1}{x^2y^2z^2} (y^2z^2 + z^2x^2 + x^2y^2) \quad \text{and} \quad Y^2Z^2 + Z^2X^2 + X^2Y^2 = \frac{1}{x^2y^2z^2} (x^2 + y^2 + z^2)$$

Hence

$$\frac{Y^2Z^2 + Z^2X^2 + X^2Y^2}{X^2 + Y^2 + Z^2} = \frac{x^2 + y^2 + z^2}{y^2z^2 + z^2x^2 + x^2y^2} = \frac{1}{t^2} = T^2$$

The following examples can be shown to be solutions to (4).

- The elementary solution $\mathcal{F}_0 = (a^2 : ab : b^2 : ab)$.
- The first solution of Fauquembergue $\mathcal{F}_1 = (ac : bc : 0 : ab)$ where $a^2 + b^2 = c^2$. 

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• The second solution of Fauquembergue $F_2 = (2a^2bc^3 : 2ab^2c^3 : (a^2 - b^2)c^4 : 2ab(a^4 + b^4))$ where $a^2 + b^2 = c^2$.

An application of the previous lemma to the second solution of Fauquembergue, we deduce a new solution to (4).

**Proposition 4.** If $a^2 + b^2 = c^2$ and
\[
\begin{align*}
x &= ac(a^2 - b^2)(a^4 + b^4) \\
y &= bc(a^2 - b^2)(a^4 + b^4) \\
z &= 2a^2b^2(a^4 + b^4) \\
t &= ab(a^2 + b^2)^2(a^2 - b^2)
\end{align*}
\]
then $(x: y: z: t)$ is a solution to (4).

We will label this solution $D_1$. For example, if $(a : b : c) = (4 : 3 : 5)$ then $(x : y : z : t) = (47180 : 35385 : 97056 : 52500)$ is a solution to (4).

### 4 An Elliptic Curve over $\mathbb{Q}(m)$

We begin this section by providing some background on elliptic surfaces, which can be defined as a one-parameter algebraic family of elliptic curves. See Silverman [5, Chapter 3].

Let $C$ be a curve defined over a field $k$. Consider a rational map $C \to \mathbb{P}^1$. The collection of all such maps is denoted by $K = k(C)$. For example, if $C : a^2 + b^2 = c^2$ is defined over $\mathbb{Q}$, we have an isomorphism $C \to \mathbb{P}^1$ given by
\[(a : b : c) \mapsto m = \frac{a}{c-b} \iff (a : b : c) = (2m : m^2 - 1 : m^2 + 1).
\]
Hence, $K = \mathbb{Q}(C) = \mathbb{Q}(m)$. This is the field we will consider.

Consider a family of curves
\[E_m : x_2^2 = x_1^3 + A(m)x_1 + B(m)\]
with rational functions $A(m), B(m) \in K$. If we rewrite our equation in homogeneous form, we form the elliptic surface
\[\mathcal{E} = \{(x_1 : x_2 : x_3, m) \in \mathbb{P}^2 \times C | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3\}.
\]
We have a map $\pi : \mathcal{E} \to C$ defined by $((x_1 : x_2 : x_3), m) \mapsto m$. For $m \in \mathbb{Q}$ such that $4A(m)^3 + 27B(m)^2 \neq 0$, the fiber
\[\mathcal{E}_m = \pi^{-1}(m) = \{(x_1 : x_2 : 1) \in \mathbb{P}^2 | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3\}
\]is the elliptic curve $E_m$ over $\mathbb{Q}$.
We say that the elliptic surface is non-split if the $j$-invariant
\[ j : C \to \mathbb{P}^1 \text{ defined by } j(m) = \frac{4A(m)^3}{4A(m)^3 + 27B(m)^2} \]
is a non-constant function. A parametrization of $E$ by $C$, or a section to $\pi$, is a map $\sigma : C \to E$ such that the composition $\pi \circ \sigma : m \mapsto m$ is the identity map on $C$. There is always a trivial section on $E$, namely the map $\sigma_0 : m \mapsto O_m = ((0 : 1 : 0), m)$. In general, the collection $E(C)$ of all sections is an abelian group, where we define
\[ \sigma_1(m) = (P(m), m) \quad \implies \quad (\sigma_1 \oplus \sigma_2)(m) = (P(m) \oplus Q(m), m). \]
\[ \sigma_2(m) = (Q(m), m) \]

We often abuse notation and write $E : x_2^2 = x_1^3 + Ax_1 + B$ as an elliptic curve over the function field $K = k(C)$. In fact, we have an isomorphism between the group of points of $E$ over $K$ and the group of sections of $E$ over $\mathbb{Q}$.

\[ E(K) \xrightarrow{\sim} E(C) \]
\[ P(m) = (x_1(m) : x_2(m) : x_3(m)) \quad \mapsto \quad \{ \sigma : m \mapsto ((x_1(m) : x_2(m) : x_3(m)), m) \} \]

It will be helpful to consider $E$ as a two-dimensional surface, where each section maps to a one-dimensional curve. The result [5, Theorem 6.1, Chapter 3] asserts that $E(K) \cong E(C)$ is a finitely generated abelian group whenever $E$ is a non-split surface.

Let $m_0 \in \mathbb{Q}$ such that $E_0 = \pi^{-1}(m_0)$ is an elliptic curve over $k = \mathbb{Q}$. Silverman’s “specialization theorem” [5, Theorem 11.4, Chapter 3] asserts that the map $E(C) \to E_0(k)$ which sends a section $\sigma : m \mapsto (P(m), m)$ to the point $P_0 = P(m_0)$ is injective for all but finitely many $m_0 \in k$. In particular, if $P_0$ is a point of finite (infinite) order in $E_0(k)$ for some $m_0 \in k$, then $P(m)$ must be a point of finite (infinite) order in $E(K)$ as a function of $m$.

Let us return to the condition
\[ (x^2 + y^2 + z^2)(y^2z^2 + z^2x^2 + x^2y^2) = \square. \]
If $a^2 + b^2 = c^2$ nonzero then we can express $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$. For the sake of space, we will leave our work below in terms of $a, b, c$. If we impose the condition $(x, y) = (a, b)$, we obtain
\[ (c^2 + z^2)(c^2z^2 + a^2b^2) = \square. \]
Dividing by $c^2$, we can consider the following equation,
\[ h^2 = z^4 + \frac{a^4 + 3a^2b^2 + b^4}{c^2} \quad z^2 + a^2b^2 \quad (7) \]
From the preceding examples $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions to (8),

\[
(z, h)_{\mathcal{F}_0} = \left( \frac{b^2}{a}, \frac{b(a^4 + a^2b^2 + b^4)}{a^2c} \right);
\]
\[
(z, h)_{\mathcal{F}_1} = (0, ab);
\]
\[
(z, h)_{\mathcal{F}_2} = \left( \frac{(a^2 - b^2)c}{2ab}, \frac{(a^4 + b^4)c^2}{4a^2b^2} \right);
\]
\[
(z, h)_{\mathcal{D}_1} = \left( \frac{2a^2b^2}{(a^2 - b^2)c}, \frac{(a^4 + b^4)ab}{(a^2 - b^2)^2} \right).
\]

**Comment 5.** If we impose the condition $(x, y) = (a, b) = (2m, m^2 - 1)$, we obtain

\[
((m^2 + 1)^2 + z^2) ((m^2 + 1)^2 z^2 + (2m)(m^2 - 1))^2 = 0.
\]

Dividing by $(m^2 + 1)^2$, we can consider the following equation,

\[
h^2 = z^4 + \frac{m^8 + 8m^6 - 2m^4 + 8m^2 + 1}{(m^2 + 1)^2} z^2 + ((2m)(m^2 - 1))^2
\]

(8)

From the preceding examples, $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions $(z, h)$ to (8), imposing the condition $a^2 + b^2 = c^2$ and $(x, y) = (a, b)$. In terms of the parameter $m$ we obtain

\[
(z, h)_{\mathcal{F}_0} = \left( \frac{(m^2 - 1)^2}{2m}, \frac{(m^2 - 1)(m^4 - 2m^3 + 2m^2 + 2m + 1)(m^4 + 2m^3 + 2m^2 - 2m + 1)}{4m^2(m^2 + 1)} \right);
\]
\[
(z, h)_{\mathcal{F}_1} = (0, 4m(m^2 - 1));
\]
\[
(z, h)_{\mathcal{F}_2} = \left( \frac{-(m^2 + 1)(m^2 + 2m - 1)(m^2 - 2m - 1)}{4m(m^2 - 1)}, \frac{(m^2 + 1)^2(m^8 - 4m^6 + 22m^4 - 4m^2 + 1)}{16m^2(m^2 - 1)^2} \right);
\]
\[
(z, h)_{\mathcal{D}_1} = \left( \frac{-8m^2(m^2 - 1)^2}{-(m^2 + 1)(m^2 + 2m - 1)(m^2 - 2m - 1)}, \frac{2m(m^2 - 1)(m^8 - 4m^6 + 22m^4 - 4m^2 + 1)}{(m^2 + 2m - 1)(m^2 - 2m - 1))^2} \right).
\]

We will show that, in fact, there are infinitely many parametric solutions to equation (4) by showing there are infinitely many parametric solutions to (8).

**Theorem 6.** Parametric solutions of the equation

\[
x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2
\]

may be obtained by finding points on an elliptic curve over the field $\mathbb{Q}(m)$. 

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Proof. By Proposition 1 and assuming \( a^2 + b^2 = c^2 \) nonzero, consider equation (8). Let 
\[
A = \frac{a^4 + 3a^2b^2 + b^4}{4(a^2 + b^2)} \quad \text{and} \quad B = \frac{a^2b^2}{4},
\]
so that (8) can be expressed as
\[
h^2 = z^4 + 4Az^2 + 4B
\]
(9)
If we have a rational solution to (9) then we get a rational solution to
\[
v^2 = u^3 + \alpha u^2 + \beta u
\]
(10)
where \( \alpha = -2A \) and \( \beta = A^2 - B \), by
\[
u = \frac{1}{2}(z^2 + 2A - h) \quad \text{and} \quad v = \frac{1}{2}z(z^2 + 2A - h).
\]
Conversely, assuming \((u, v) \neq (0, 0)\), a rational solution to (10) leads to a rational solution to (9) by
\[
z = \frac{v}{u} \quad \text{and} \quad h = \frac{v^2}{u^2} + 2A - 2u.
\]
The discriminant of (10) is
\[
(\alpha^2 - 4\beta)^2 = 4B(A^2 - B)^2 = \frac{a^2b^2(a^4 + a^2b^2 + b^4)^4}{256(a^2 + b^2)^4}
\]
which is nonzero since at least one of \( a, b \) are nonzero. Hence (10) defines an elliptic curve
over the field \( \mathbb{Q}(m) \) and every point \((u, v)\) on this elliptic curve yields a point \((z, h)\) satisfying
(8), which implies a solution \((x, y, z, t)\) to (4).

We will show how to obtain new solutions, by adding points on the elliptic curve. We write \( E \) for the elliptic curve (10) over \( \mathbb{Q}(m) \), + for the addition of points on the curve (10), and \( P_S \) will denote a point on \( E \) yielding a solution \( S \) to (4). The addition of points on an elliptic curve is described in Silverman [6]. Note that instead of writing \( P + P \) we will write \( 2P \).

Example 7. The solutions \( \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \) and \( \mathcal{D}_1 \) to (4) are provided by the following points on (10):
\[
P_{\mathcal{F}_0} = \left( \frac{(c - b)(a^2 - ab + b^2)}{4(c + b)} \frac{(a^2 + ab + b^2)}{4(c + b)} , \frac{b^2(c - b)(a^2 - ab + b^2)}{4a(c + b)(a^2 + b^2)} \right)
\]
\[
P_{\mathcal{F}_1} = \left( \frac{(a^2 - ab + b^2)^2}{4(a^2 + b^2)} , 0 \right)
\]
\[
P_{\mathcal{F}_2} = \left( \frac{a^2b^2}{4(a^2 + b^2)} , \frac{ab(a^2 - b^2)}{8c} \right)
\]
\[
P_{\mathcal{D}_1} = \left( \frac{(a - b)(a^2 - ab + b^2)^2}{4(a + b)^2(a^2 + b^2)} , \frac{a^2b^2(a - b)(a^2 + ab + b^2)^2}{2c(a + b)^3(a^2 + b^2)} \right)
\]
Let us remind the reader of the Lutz-Nagell theorem, which will be used in the proof of Theorem 8 along with the “specialization theorem”.

**Theorem 8.** Let \( E \) be given by \( y^2 = x^3 + Ax + B \) with \( A, B \in \mathbb{Z} \). Let \( P = (x, y) \in E(\mathbb{Q}) \). Suppose \( P \) has finite order. Then \( x, y \in \mathbb{Z} \). If \( y \neq 0 \) then \( y^2 \) divides \( 4A^3 + 27B^2 \).

**Theorem 9.** There exists infinitely many points on \((10)\).

**Proof.** Let \( N = \frac{a^2 - ab + b^2}{2c} \) and \( L = \frac{a^2 + ab + b^2}{2c} \). Then \((10)\) can be expressed as

\[
E : v^2 = u \left( u - N^2 \right) \left( u - L^2 \right)
\]

(11)

It can be shown that the rank of this elliptic curve over \( \mathbb{Q}(m) \) is at least one. To show the rank is at least one, specialize at say, \((a, b) = (3, 4)\). Then the point \( P = (36/25, 21/10) \) is on the curve

\[
E_1 : v^2 = u^3 - \frac{769}{50}u^2 + \frac{231361}{10000}u.
\]

In order to use the Lutz-Nagell theorem, we need to express \( E_1 \) in Weierstrass form with integral coefficients:

\[
E'_1 : y^2 = x^3 - 1538x^2 + 231361x.
\]

The point \( P \) on \( E_1 \) corresponds to \( P' = (144, 2100) \) on \( E'_1 \), and thus

\[
2P' = (70980625/28224, 389867877575/4741632).
\]

Since \( 2P' \) is not an integral point, \( P' \) is not a point of finite order on \( E'_1 \), so by the “Specialization Theorem” the rank of \((11)\) is of positive rank, these are the same ideas as in Ulas [7]. Also note, calculations found at least 17 distinct points on \( E \). The maximum number of torsion points is 16, so the rank must be at least one.

**Remark 10.** On \( E \), the torsion points are \((0,0)\), \((N^2,0)\), \((L^2,0)\), and \( O \), the point at infinity.

**Theorem 11.** Every solution to the equation

\[
x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2
\]

such that \((x, y) = (a, b)\), \( a \) and \( b \) nonzero, proceeds from exactly two points on \( E \) different from \((0,0)\). If one of them is \( P = (u, v) \), with \( u \neq 0 \), then the other one is \( P' = (u', v') = \lambda(u, v) \), with \( \lambda = \frac{\beta}{u^2} \).

**Proof.** Although \( E \) is defined over a function field, Figure 2 provides some intuition for our proof. Let \( P = (u, v) \) be a point on \( E \), with \( u \neq 0 \), which yields a solution to \((4)\). If there exists a point \( P' = (u', v') \) on \( E \) different from \((0,0)\) yielding the same solution to \((4)\) as
Figure 2: \( E : v^2 = u^3 + \alpha u^2 + \beta u \)

\((u, v)\), then \( z' = z \) so \( u' \neq 0 \) and \( \frac{v'}{u'} = \frac{v}{u} \). Thus there exists a rational \( \lambda \) such that \( u' = \lambda u \) and \( v' = \lambda v \) and

\[
0 = -v'^2 + u'^3 + \alpha u'^2 + \beta u' = -\lambda^2 v'^2 + \lambda^3 u'^3 + \alpha \lambda^2 u'^2 + \beta \lambda u.
\]

By substituting for \( v'^2 \), we have

\[
0 = -\lambda^2(u'^3 + \alpha u'^2 + \beta u) + \lambda^3 u'^3 + \alpha \lambda^2 u'^2 + \beta \lambda u = \lambda (\lambda - 1) u (\lambda u^2 - \beta).
\]

Since \( u' \neq 0 \), then \( \lambda \neq 0 \). Assuming \((u', v') \neq (u, v)\) implies \( \lambda \neq 1 \). Thus \( \lambda = \frac{\beta}{u^2} \).

To show \((u', v')\) is on \( E \), notice

\[
u'^3 + \alpha u'^2 + \beta u' = \frac{\beta^3}{u^3} + \alpha \frac{\beta^2}{u^2} + \frac{\beta^2}{u} = \frac{\beta^2}{u^2} (\beta u + \alpha u^2 + u^3) = \frac{\beta^2}{u^2} v^2 = \left(\frac{\beta}{u^2} v\right)^2 = v'^2.
\]

Both of these points yield the same solution to (4) since \( x = a, y = b \) and \( z = z' \). This defines a solution \((x : y : z : t)\), except possibly for the sign of \( t \).

Remark 12. If \( P_{\mathcal{F}_2} = (u, v) \), we find that \( 2P_{\mathcal{F}_0} = (u', v') \), both yielding the same solution \( \mathcal{F}_2 \).

From the previous two theorems we deduce the following corollary.

Corollary 13. There exists infinitely many parametric solutions to \( x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \).
5 Obtaining new parametric solutions

Before obtaining new parametric solutions, we can interpret Lemma 3 in terms of points in $E(K)$ where $K = \mathbb{Q}(m)$ and $E$ as described earlier.

**Proposition 14.** Let $(u, v) \in E(K)$ such that $(u, v) \notin \{O, (0, 0), (N^2, 0), (L^2, 0)\}$, and let $(u', v'), (u'', v'') \in E(K)$ such that:

$$(u, v) + (u', v') = (N^2, 0) \quad \text{and} \quad (u, v) + (u'', v'') = (L^2, 0).$$

If $(u, v)$ yields a solution $(x : y : z : t)$ to (4) such that $xyzt \neq 0$, then $(u', v')$ and $(u'', v'')$ both yield $\left(\frac{1}{y} : \frac{1}{x} : \frac{1}{z} : \frac{1}{t}\right)$, except perhaps the signs of $\frac{1}{z}$ and $\frac{1}{t}$.

![Figure 3: $E : v^2 = u^3 + \alpha u^2 + \beta u$](image)

**Proof.** From the group law on equation (10), we deduce

$$(u', v') = \left(\frac{N^2(u - L^2)}{u - N^2}, \frac{N^2(N^2 - L^2)v}{(u - N^2)^2}\right), \quad (u'', v'') = \left(\frac{L^2(u - N^2)}{u - L^2}, \frac{L^2(L^2 - N^2)v}{(u - L^2)^2}\right).$$

If $(u', v')$ yields $(x' : y' : z' : t')$ then $z' = \frac{v'}{u'} = \frac{(N^2 - L^2)v}{(u - N^2)(u - L^2)}$. From the relations $N^2 - L^2 = -ab$ and $v^2 = u(u - N^2)(u - L^2)$, we conclude

$$z' = -ab \frac{u}{v} = -\frac{ab}{z}. $$
Thus if \((u, v)\) yields \((x : y : z : t) = (a : b : z : t)\), then \((u', v')\) yields

\[
(x' : y' : z' : t') = \left( a : b : \frac{ab}{z} : \frac{ab}{t} \right) = \left( \frac{1}{b} : \frac{1}{a} : \frac{1}{z} : \frac{1}{t} \right) = \left( \frac{1}{y} : \frac{1}{x} : -\frac{1}{z} : \frac{1}{t} \right),
\]

where the signs of \(z'\) and \(t'\) may be positive or negative. The proof for \(z''\) is similar.

Next let \(P_{D_2} = P_{F_0} + P_{D_1}\). If \(a^2 + b^2 = c^2\), we find \(P_{D_2} = (u, v)\) with

\[
u = \frac{(c + a)(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)^2}{4(c - a)(a^2 + b^2)(2a^3 + a^2c - b^2c)^2}, \quad v = \frac{a^2b(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)(2b^3 - a^2c + b^2c)(2b^3 + a^2c - b^2c)}{4(c - a)^2(a^2 + b^2)(2a^3 + a^2c - b^2c)^3}
\]

By the same methods used in the proof of Theorem 6, we deduce the following solution \(D_2\):

**Proposition 15.** If \(a^2 + b^2 = c^2\) and if

\[
x = ab(2a^3 - a^2c + b^2c)(2a^3 + a^2c - b^2c)(2ab^2 + a^2c + b^2c)(-2ab^2 + a^2c + b^2c) \\
y = b^2(2a^3 - a^2c + b^2c)(2a^3 + a^2c - b^2c)(2ab^2 + a^2c + b^2c)(-2ab^2 + a^2c + b^2c) \\
z = a^3(2b^3 - a^2c + b^2c)(2b^3 + a^2c - b^2c)(2ab^2 + a^2c + b^2c)(-2ab^2 + a^2c + b^2c) \\
t = ab(2a^3 - a^2c + b^2c)(2a^3 + a^2c - b^2c)(2a^2b + a^2c + b^2c)(-2a^2b + a^2c + b^2c)
\]

then \(D_2 = (x : y : z : t)\) is a solution to (4).

**Example 16.** Since \((a : b : c) = (2m : m^2 - 1 : m^2 + 1)\), if \(m = 2\), then

\[
x = 1 899 301 428 \\
y = 1 424 476 071 \\
z = 282 491 696 \\
t = 1 165 848 372
\]

Next let \(P_{D_3} = 2P_{D_1}\). We find that \(P_{D_3} = (u, v)\) with:

\[
u = \frac{(a^2 + b^2)(a^4 + b^4)^2}{16a^2b^2(a - b)^2(a + b)^2}, \quad v = \frac{PQRS(a^4 + b^4)}{64a^3b^3c(a - b)^3(a + b)^3}
\]

where

\[
P = -a^3 + a^2b + ab^2 + b^3, \quad Q = a^3 - a^2b + ab^2 + b^3 \\
R = a^3 + a^2b - ab^2 + b^3, \quad S = a^3 + a^2b + ab^2 - b^3.
\]

From this we deduce \(D_3\):
Proposition 17. If \(a^2 + b^2 = c^2\) and if
\[
\begin{align*}
  x &= \frac{a^2bc(a^2 - b^2)(a^2 + b^2)(a^4 + b^4)}{4(a^2 + b^2)^2Q^2R^2}, \\
  y &= \frac{ab^2c(a^2 - b^2)(a^2 + b^2)(a^4 + b^4)}{4(a^2 + b^2)^2Q^2R^2}, \\
  z &= PQRSG \\
  t &= 4ab(a^2 - b^2)(a^2 + b^2)^2(a^4 + b^4)H
\end{align*}
\]
with \(P, Q, R, S\) as above, and
\[
\begin{align*}
  G &= a^{12} + 6a^{10}b^2 - a^4b^8 + 4a^6b^6 - a^4b^8 + 6a^2b^{10} + b^{12} \\
  H &= a^{12} - 2a^{10}b^2 + 7a^8b^4 + 4a^6b^6 + 7a^4b^8 - 2a^2b^{10} + b^{12},
\end{align*}
\]
then \(D_2 = (x : y : z : t)\) is a solution to \((4)\).

Now put \(P_{D_4} = P_{D_3} + P_{D_1}\). We find \(P_{D_4} = (u, v)\), with
\[
\begin{align*}
  u &= \frac{(a^2 - ab + b^2)^2P^2S^2}{4(a^2 + b^2)^2Q^2R^2}, \\
  v &= \frac{a^2b^2c(a^2 - b^2)(a^2 - ab + b^2)^2(a^4 + b^4)PS}{Q^3R^3}
\end{align*}
\]
From this we deduce the following solution \(D_4:\)

Proposition 18. If \(a^2 + b^2 = c^2\) and if
\[
\begin{align*}
  x &= acPQRSH \\
  y &= bcPQRSH \\
  z &= 4a^2b^2(a^2 - b^2)(a^2 + b^2)^2(a^4 + b^4)H \\
  t &= abPQRSG
\end{align*}
\]
with the same \(P, Q, R, S, G, H\) as in Proposition 17, then \(D_4 = (x : y : z : t)\) is a solution to \((4)\).

The degree of this solution is 26 in \((a : b : c)\), and hence 52 in the homogeneous coordinates \((m : n)\) if we express \((a : b : c) = (2mn : m^2 - n^2 : m^2 + n^2)\) for nonzero integers \(m, n\).

Remark 19. The values \(z\) for \(P_{D_4}\) and \(z'\) for \(P_{D_4}\) satisfy \(zz' = ab\), so except perhaps exchanging \((x', y')\) and \((y', x')\), \(D_4\) is deduced from \(D_3\) by replacing \((x, y, z)\) by \(\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\).

In summary, parametric solutions \((x : y : z : t)\) to \((4)\) with their degree are shown in Table 1.

<table>
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<th>solution</th>
<th>(F_0)</th>
<th>(F_1)</th>
<th>(F_2)</th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
</tr>
</thead>
<tbody>
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<td>4</td>
<td>12</td>
<td>16</td>
<td>28</td>
<td>48</td>
<td>52</td>
</tr>
</tbody>
</table>

Table 1: Degree of parametric solutions
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References


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