Infinite Products Involving $\zeta(3)$ and Catalan’s Constant

Yasuyuki Kachi  
Department of Mathematics  
University of Kansas  
Lawrence, KS 66045-7523  
USA  
kachi@math.ku.edu

Pavlos Tzermias  
Department of Mathematics  
University of Patras  
26500 Rion (Patras)  
Greece  
tzermias@math.upatras.gr

Abstract

We present some infinite product formulas for $e^{\frac{7\zeta(3)}{\pi^2}}$, $e^{\frac{4G}{\pi}}$ and $e^{\frac{2G}{\pi} \pm \frac{1}{2}}$, where $G$ is Catalan’s constant. We relate these formulas to similar ones obtained by Guillera and Sondow in the context of their systematic study of Lerch’s transcendent. Our proofs are entirely elementary.

1 Introduction

This paper studies some infinite product formulas involving two classical constants, namely $\zeta(3)$ and Catalan’s constant, whose definition we now recall:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$
and
\[ G = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}. \]

The following formulas are reminiscent of similar formulas obtained by Guillera and Sondow in [5]:

**Proposition 1.** The following formulas hold:

\[ e^{\frac{\zeta(3)}{4\pi^2} + \frac{1}{4}} = \lim_{m \to \infty} \prod_{n=1}^{2m+1} \frac{1}{\sqrt[n]{e}} \left(1 - \frac{1}{n+1}\right)^{\frac{n(n+1)}{2}}(-1)^n. \] (1)

\[ e^{\frac{\zeta(3)}{4\pi^2} - \frac{1}{4}} = \lim_{m \to \infty} \prod_{n=1}^{2m} \sqrt[n]{e} \left(1 - \frac{1}{n+1}\right)^{\frac{n(n+1)}{2}}(-1)^n. \] (2)

\[ e^{\frac{\zeta(3)}{4\pi^2}} = \lim_{m \to \infty} \left(\frac{2^{22} \cdot 4^{42} \cdot 6^{62} \cdots (2m)^{(2m)^2}}{11^{12} \cdot 3^{32} \cdot 5^{52} \cdots (2m-1)^{(2m-1)^2}}\right)^4 \left(\frac{(2m+2)^{4m+5}}{(2m+1)^{12m+9}}\right)^m. \] (3)

**Proposition 2.** The following formulas hold:

\[ e^{\frac{G}{\pi} - \frac{1}{2}} = \lim_{m \to \infty} \prod_{n=1}^{2m} \left(1 - \frac{2}{2n+1}\right)^{n(-1)^n}. \] (4)

\[ e^{\frac{G}{\pi} + \frac{1}{2}} = \lim_{m \to \infty} \prod_{n=1}^{2m+1} \left(1 - \frac{2}{2n+1}\right)^{n(-1)^n}. \] (5)

\[ e^{\frac{4G}{\pi}} = \lim_{m \to \infty} \left(\frac{3^3 \cdot 7^7 \cdot 11^{11} \cdots (4m-1)^{4m-1}}{11^1 \cdot 5^5 \cdot 9^9 \cdots (4m-3)^{4m-3}}\right)^2 \frac{(4m+3)^{2m+1}}{(4m+1)^{6m+1}}. \] (6)

We claim no novelty for the formulas themselves; our only purpose here is to present completely elementary proofs of these formulas and to establish the not-so-obvious facts below:

**Fact 3.** Formula (3) is equivalent to the following formula given by Guillera and Sondow [5, Example 5.3]:

\[ e^{\frac{\zeta(3)}{4\pi^2}} = e^{\sum_{n=1}^{\infty} \frac{n(n+1)}{2n+3} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \log(k+1)} \]

\[ = \prod_{n=1}^{\infty} \left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1} \binom{n}{k} \frac{n(n+1)}{2n+3}}\right) \]

\[ = \left(\frac{2^1}{1^1}\right)^{\frac{1^2}{2^2}} \left(\frac{2^2}{1^1 \cdot 3^1}\right)^{\frac{2^3}{2^3}} \left(\frac{2^3 \cdot 4^1}{1^1 \cdot 3^3}\right)^{\frac{3^4}{3^4}} \left(\frac{2^4 \cdot 4^4}{1^1 \cdot 3^6 \cdot 5^1}\right)^{\frac{4^5}{5^5}} \cdots . \]
Fact 4. Formula (4) follows from rearranging the factors of the following formula given by Guillera and Sondow [5, Example 5.5]:

\[ e^{\frac{G}{\pi}} = e^{\sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^k} \right) \log(2k+1)} \]

\[ = \prod_{n=1}^{\infty} \left( \prod_{k=0}^{n} \left( 2k + 1 \right) \right)^{(-1)^{k+1} \left( \binom{n}{k} \right)} \]

\[ = \left( \frac{3^1}{1!} \right)^{\frac{1}{\pi}} \left( \frac{3^2}{1^1 \cdot 5^1} \right)^{\frac{2}{\pi}} \left( \frac{3^3 \cdot 7^1}{1^1 \cdot 5^3} \right)^{\frac{3}{\pi}} \left( \frac{3^4 \cdot 7^4}{1^1 \cdot 5^6 \cdot 9^1} \right)^{\frac{4}{\pi}} \cdots, \]

which in turn is equivalent to formula (6).

2 Proof of Proposition 2

We begin with the following formula which is a classically known Fourier expansion (see, for example, Exercise 11.15(c) in [1, p. 338]):

Formula 5. Let \( \sigma \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \{0\} \). Then

\[ \sum_{m=0}^{\infty} \cos \left( \frac{\pi (2m + 1) \sigma}{2m + 1} \right) = \frac{1}{2} \log \left| \cot \left( \frac{\pi}{2} \sigma \right) \right|. \]

The following formula, which follows directly from Formula 5 by integrating both sides over the interval \( [0, \frac{1}{2}] \), is also well-known (see, for example, [2, p. 239]):

Formula 6.

\[ G = \int_{\theta=0}^{\pi/4} \log \left( \cot \theta \right) d\theta. \]

By applying integration by parts to the latter integral, we obtain

Corollary 7.

\[ G = \frac{1}{2} \int_{\alpha=0}^{1/2} \frac{\pi^2 \alpha}{\sin(\pi \alpha)} d\alpha. \]

The following formula is also well-known (see, for example, [8, p. 155]):

Formula 8. Let \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \) and \( s \in \left( -\frac{1}{2}, \frac{1}{2} \right) \). Then

\[ \cos(2\pi \alpha s) = \frac{\sin(\pi \alpha)}{\pi} \left( \frac{1}{\alpha} + 2\alpha \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha^2 - m^2} \cos(2\pi ms) \right). \]

Setting \( s = 0 \) in Formula 8 gives:
Corollary 9. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Then

$$1 = \frac{\sin(\pi \alpha)}{\pi} \left( \frac{1}{\alpha} + 2\alpha \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha^2 - m^2} \right).$$

Lemma 10. Let $m \in \mathbb{Z}$, $m \geq 1$. Then

$$\int_{\alpha=0}^{1/2} \frac{\alpha^2}{\alpha^2 - m^2} \, d\alpha = \frac{1}{2} + \frac{m}{2} \log \frac{2m-1}{2m+1}.$$

Proof. This is straightforward:

$$\int_{\alpha=0}^{1/2} \frac{\alpha^2}{\alpha^2 - m^2} \, d\alpha = \frac{1}{2} \int_{\alpha=0}^{1/2} \left( 2 + \frac{m}{\alpha - m} - \frac{m}{\alpha + m} \right) \, d\alpha$$

$$= \frac{1}{2} \left[ 2\alpha + m \log(-\alpha + m) - m \log(\alpha + m) \right]_{\alpha=0}^{1/2}$$

$$= \frac{1}{2} + \frac{m}{2} \log \frac{2m-1}{2m+1}.$$ 

We now proceed with the proof of formula (4). By Corollary 9, we have

$$\frac{\pi^2 \alpha}{\sin(\pi \alpha)} = \pi + 2\pi \alpha^2 \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha^2 - m^2}.$$

Integrating both sides with respect to $\alpha$ over the interval $[0, \frac{1}{2}]$ gives

$$\int_{\alpha=0}^{1/2} \frac{\pi^2 \alpha}{\sin(\pi \alpha)} \, d\alpha = \frac{\pi}{2} + 2\pi \int_{\alpha=0}^{1/2} \left( \sum_{m=1}^{\infty} \frac{(-1)^m \alpha^2}{\alpha^2 - m^2} \right) \, d\alpha. \quad (7)$$

Consider the sequence of functions

$$f_m(\alpha) = \frac{(-1)^m \alpha^2}{\alpha^2 - m^2}$$

on the interval $I = [0, \frac{1}{2}]$, where $m = 1, 2, \ldots$. Since $\alpha \in I$, we clearly have

$$|f_m(\alpha)| = \frac{\alpha^2}{|\alpha^2 - m^2|} \leq \frac{\frac{1}{4}}{m^2 - \frac{1}{4}} = \frac{1}{4m^2 - 1} \leq \frac{1}{2m^2},$$

for all $m$. Since

$$\sum_{m=1}^{\infty} \frac{1}{2m^2}.$$
converges, it follows from the Weierstrass $M$-test that the series
\[ \sum_{m=1}^{\infty} f_m(\alpha) \]
converges uniformly on $I$, and, by well-known principles, (see, for example, [1, Thm. 9.9, p. 226]), can therefore be integrated term by term. In other words, if we set
\[ a_m = (-1)^m \left( 1 + m \log \frac{2m - 1}{2m + 1} \right), \]
then (7) and Lemma 10 imply that
\[ \int_{\alpha=0}^{1/2} \frac{\pi^2 \alpha}{\sin(\pi \alpha)} \, d\alpha = \frac{\pi}{2} + \pi \sum_{m=1}^{\infty} a_m. \tag{8} \]
The left-hand side of (8) is a definite integral of the continuous function $\frac{\pi^2 \alpha}{\sin(\pi \alpha)}$ over the interval $[0, \frac{1}{2}]$. Hence the left-hand side of (8) is a real number which implies that
\[ \lim_{m \to \infty} a_m = 0. \]
Keeping this in mind, define
\[ A_n = \sum_{m=1}^{n} a_m. \]
Then, by (8), we have
\[
\int_{\alpha=0}^{1/2} \frac{\pi^2 \alpha}{\sin(\pi \alpha)} \, d\alpha = \frac{\pi}{2} + \pi \lim_{n \to \infty} A_n = \frac{\pi}{2} + \pi \lim_{N \to \infty} A_{2N} = \frac{\pi}{2} + \pi \lim_{N \to \infty} \sum_{m=1}^{N} (a_{2m-1} + a_{2m})
\]
\[
= \pi \left( \frac{1}{2} + \lim_{N \to \infty} \sum_{m=1}^{N} \left( - (2m - 1) \log \frac{4m - 3}{4m - 1} + 2m \log \frac{4m - 1}{4m + 1} \right) \right)
\]
\[
= \pi \left( \frac{1}{2} + \lim_{N \to \infty} \log \prod_{m=1}^{N} \frac{(4m - 1)^{4m-1}}{(4m - 3)^{2m-1}(4m + 1)^{2m}} \right)
\]
\[
= \pi \left( \frac{1}{2} + \log \prod_{m=1}^{\infty} \frac{(4m - 1)^{4m-1}}{(4m - 3)^{2m-1}(4m + 1)^{2m}} \right).
\]
By Corollary 7, the left-hand side equals $2G$, therefore
\[ \frac{2G}{\pi} - \frac{1}{2} = \log \prod_{m=1}^{\infty} \frac{(4m - 1)^{4m-1}}{(4m - 3)^{2m-1}(4m + 1)^{2m}}. \]
Therefore,
\[ e^{\frac{2\pi}{1 - \frac{1}{2}}} = \lim_{m \to \infty} \frac{3^3}{1 \cdot 5^2} \cdot \frac{7^7}{5^3 \cdot 9^4} \cdot \frac{11^{11}}{9^5 \cdot 13^6} \cdots \frac{(4m - 1)^{4m-1}}{(4m - 3)^{2m-1}(4m + 1)^{2m}} \]

\[ = \lim_{m \to \infty} \frac{3^3 \cdot 7^7 \cdot 11^{11} \cdots (4m - 1)^{4m-1}}{5^5 \cdot 9^9 \cdot 13^{13} \cdots (4m - 3)^{4m-3} \cdot (4m + 1)^{2m}} \]

\[ = \lim_{m \to \infty} \left( \frac{1}{3} \right)^{-1} \left( \frac{3}{5} \right)^2 \left( \frac{5}{7} \right)^{-3} \cdots \left( \frac{4m - 1}{4m + 1} \right)^{2m} = \lim_{m \to \infty} \prod_{n=1}^{2m} \left( 1 - \frac{2}{2n + 1} \right)^{n(-1)^n} , \]

and this completes the proof of formula (4). Multiplying both sides of the latter formula by \(e\) and using the fact that

\[ e = \lim_{m \to \infty} \left( 1 - \frac{2}{4m + 3} \right)^{-2m+1} \]

gives formula (5). Finally, multiplying formulas (4) and (5) together and expanding gives formula (6).

### 3 Proof of Proposition 1

We will first prove formula (1).

**Lemma 11.** Let \( m \in \mathbb{N} \) and \( \delta \in \left( 0, \frac{1}{2} \right) \). Then

\[ \pi^2 \int_{\sigma=\delta}^{1/2} \left( \frac{1}{2} - \sigma \right) \frac{\cos \left( \pi(2m+1)\sigma \right)}{2m+1} \, d\sigma = \frac{\cos \left( \pi(2m+1)\delta \right)}{(2m+1)^3} + \frac{\pi \left( \delta - \frac{1}{2} \right) \sin \left( \pi(2m+1)\delta \right)}{(2m+1)^2}. \]

**Proof.** This is straightforward integration by parts:

\[ \int_{\sigma=\delta}^{1/2} \left( \frac{1}{2} - \sigma \right) \frac{\cos \left( \pi(2m+1)\sigma \right)}{2m+1} \, d\sigma = \left[ \left( \frac{1}{2} - \sigma \right) \frac{\sin \left( \pi(2m+1)\sigma \right)}{\pi(2m+1)^2} \right]_{\sigma=\delta}^{1/2} + \int_{\sigma=\delta}^{1/2} \frac{\sin \left( \pi(2m+1)\sigma \right)}{\pi(2m+1)^2} \, d\sigma \]

\[ = \left( \delta - \frac{1}{2} \right) \frac{\sin \left( \pi(2m+1)\delta \right)}{\pi(2m+1)^2} - \left[ \frac{\cos \left( \pi(2m+1)\sigma \right)}{\pi^2(2m+1)^3} \right]_{\sigma=\delta}^{1/2}, \]

and the claim follows. \( \square \)

**Corollary 12.** Let \( m \in \mathbb{N} \). Then

\[ \pi^2 \int_{\sigma=0}^{1/2} \left( \frac{1}{2} - \sigma \right) \frac{\cos \left( \pi(2m+1)\sigma \right)}{2m+1} \, d\sigma = \frac{1}{(2m+1)^3}. \]
Proof. In Lemma 11, let $\delta \to 0+$. We now recall the following basic formula:

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} = \frac{7}{8} \zeta(3). \quad (9)$$

We will establish the following:

**Formula 13.**

$$\zeta(3) = \frac{4}{7} \pi G - \frac{2}{7} \pi^2 \int_{\sigma=0}^{1/2} \frac{\pi \sigma^2}{\sin(\pi \sigma)} \, d\sigma. \quad (10)$$

Proof. First, we may rewrite Formula 6 as

$$G = \int_{\sigma=0}^{1/2} \frac{\pi}{2} \log \left( \cot \left( \frac{\pi}{2} \sigma \right) \right) \, d\sigma. \quad (10)$$

Second, by (9) and Corollary 12, we have

$$\frac{7}{8} \zeta(3) = \sum_{m=0}^{\infty} \pi^2 \int_{\sigma=0}^{1/2} \left( -\sigma \right) \cos \left( \frac{\pi(2m+1)\sigma}{2m+1} \right) \, d\sigma. \quad (11)$$

Fix $\delta \in \left(0, \frac{1}{2}\right)$. For each $n \in \mathbb{N}$, define the function

$$F_n(\sigma) = \sum_{m=0}^{n} \left( \frac{1}{2} - \sigma \right) \cos \left( \frac{\pi(2m+1)\sigma}{2m+1} \right)$$

on the interval $I = \left[\delta, \frac{1}{2}\right]$. The sequence

$$\left\{ \sum_{m=0}^{n} \cos \left( \frac{\pi(2m+1)\sigma}{2m+1} \right) \right\}_{n \in \mathbb{N}}$$

of functions is uniformly bounded on $I$ by $(2\sin(\pi\delta))^{-1}$ (see [1, Formula (15), p. 198] or [6, Item 185.5, p. 316]), whereas the sequence

$$\left\{ \left( \frac{1}{2} - \sigma \right) \frac{1}{2m+1} \right\}_{m \in \mathbb{N}}$$

clearly tends monotonically to 0 uniformly on $I$. Hence by applying Dirichlet’s test for uniform convergence (see [1, Thm. 9.15, p. 230] or [6, p. 347]), it follows that the sequence of functions $F_n(\sigma)$ converges uniformly on $I$. Therefore, the series

$$\sum_{m=0}^{\infty} \left( \frac{1}{2} - \sigma \right) \frac{\cos \left( \frac{\pi(2m+1)\sigma}{2m+1} \right)}{2m+1}$$

can be integrated term by term on $I$. Hence, Lemma 11 establishes the following
Formula 14.

\[
\pi^2 \int_{\sigma=\delta}^{1/2} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2} - \sigma\right) \cos\left(\frac{\pi(2m+1)\sigma}{2m+1}\right)}{2m+1} \ d\sigma
\]

\[
= \sum_{m=0}^{\infty} \frac{\cos\left(\frac{\pi(2m+1)\delta}{(2m+1)^3}\right)}{(2m+1)} + \pi \left(\delta - \frac{1}{2}\right) \sum_{m=0}^{\infty} \frac{\sin\left(\frac{\pi(2m+1)\delta}{(2m+1)^2}\right)}{(2m+1)^2}.
\]

Now take the limits of both sides of the latter formula as \(\delta \to 0^+\). By the Weierstrass \(M\)-test, both series on the right-hand side of Formula 14 are uniformly convergent series of functions of \(\delta\) on the interval \(I = [\delta, \frac{1}{2}]\). Therefore, we can interchange limits and infinite sums on the right-hand side of Formula 14 (see [1, Thm. 9.7, p. 220]). By (11), it follows that

\[
\pi^2 \int_{\sigma=0}^{1/2} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2} - \sigma\right) \cos\left(\frac{\pi(2m+1)\sigma}{2m+1}\right)}{2m+1} = \frac{7}{8} \zeta(3). \quad (12)
\]

Combining (10), (12) and Formula 5 gives

\[
\frac{7}{8} \zeta(3) = \pi^2 \int_{\sigma=0}^{1/2} \left(\frac{1}{2} - \sigma\right) \frac{1}{2} \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma
\]

\[
= \frac{\pi^2}{4} \left(\int_{\sigma=0}^{1/2} \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma - \int_{\sigma=0}^{1/2} 2\sigma \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma\right)
\]

\[
= \frac{\pi}{2} G - \frac{\pi^2}{2} \int_{\sigma=0}^{1/2} \sigma \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma.
\]

In short,

\[
\zeta(3) = \frac{4}{\pi} \pi G - \frac{4}{\pi} \pi^2 \int_{\sigma=0}^{1/2} \sigma \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma. \quad (13)
\]

Formula 13 now follows because

\[
\int_{\sigma=0}^{1/2} \sigma \log\left(\cot\left(\frac{\pi}{2}\right)\right) d\sigma
\]

\[
= \left[\frac{\sigma^2}{2} \log\left(\cot\left(\frac{\pi}{2}\right)\right)\right]_{\sigma=0}^{1/2} - \int_{\sigma=0}^{1/2} \frac{\sigma^2}{2} \frac{1}{\cot\left(\frac{\pi}{2}\right)} \frac{-1}{\sin\left(\frac{\pi}{2}\right)} \frac{\pi}{2} d\sigma
\]

\[
= 0 + \int_{\sigma=0}^{1/2} \frac{\sigma^2}{2} \frac{1}{\sin\left(\frac{\pi}{2}\right)} \frac{\pi}{2} d\sigma = \frac{1}{2} \int_{\sigma=0}^{1/2} \frac{\pi\sigma^2}{\sin\left(\frac{\pi}{2}\right)} d\sigma.
\]

\(\Box\)

The following statement is similar to Lemma 10.
Lemma 15. Let $m \in \mathbb{Z}$, $m \geq 1$. Then

$$\int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 - m^2} \, d\sigma = \frac{1}{8} + \frac{m^2}{2} \log \frac{4m^2 - 1}{4m^2}.$$  

Proof. This is straightforward:

$$\int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 - m^2} \, d\sigma = \left[ \frac{1}{2} \sigma^2 + m^2 \frac{1}{2} \log (-\sigma^2 + m^2) \right]_{\sigma=0}^{1/2}$$

$$= \frac{1}{8} + \frac{m^2}{2} \log \frac{m^2 - \frac{1}{4}}{m^2}. \quad \square$$

Lemma 16.

$$\lim_{n \to \infty} \frac{e^n}{\left(1 + \frac{1}{n}\right)^n} = \lim_{n \to \infty} \frac{e^{-n}}{\left(1 - \frac{1}{n}\right)^n} = \sqrt{e}.$$  

Proof. By taking logarithms, it suffices to show that

$$\lim_{n \to \infty} \left( n - n^2 \log \left(1 + \frac{1}{n}\right) \right) = \frac{1}{2} = \lim_{n \to \infty} \left( -n - n^2 \log \left(1 - \frac{1}{n}\right) \right).$$

This follows by substituting $x = \pm \frac{1}{n}$ in the Maclaurin series of the function $\log (1 + x)$ and using continuity. \quad \square

We now proceed with the proof of formula (1). By Corollary 9, we have

$$\frac{\pi \sigma^2}{\sin (\pi \sigma)} = \sigma + 2\sigma^3 \sum_{m=1}^{\infty} \frac{(-1)^m}{\sigma^2 - m^2}.$$  

Integrating both sides with respect to $\sigma$ over the interval $\left[0, \frac{1}{2}\right]$ and using formula (4) (and its proof) and Lemma 15 gives
\[
\int_{\sigma=0}^{1/2} \frac{\pi \sigma^2}{\sin (\pi \sigma)} \, d\sigma = \int_{\sigma=0}^{1/2} \left( \sigma + 2 \sigma^3 \sum_{m=1}^{\infty} \frac{(-1)^m}{\sigma^2 - m^2} \right) \, d\sigma
\]
\[
= \int_{\sigma=0}^{1/2} \sigma \, d\sigma + 2 \int_{\sigma=0}^{1/2} \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{\sigma^2 - m^2} \right) \, d\sigma
\]
\[
= \frac{1}{8} + 2 \sum_{m=1}^{\infty} (-1)^m \int_{\sigma=0}^{1/2} \frac{\sigma^3}{\sigma^2 - m^2} \, d\sigma
\]
\[
= \frac{1}{8} + 2 \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{8} + \frac{m^2}{2} \log \frac{4m^2 - 1}{4m^2} \right)
\]
\[
= \frac{1}{8} + \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{4} + m^2 \log \frac{4m^2 - 1}{4m^2} \right),
\]
which equals
\[
\frac{1}{8} + \sum_{\ell=1}^{\infty} \left( - \left( \frac{1}{4} + (2\ell - 1)^2 \log \frac{4(2\ell - 1)^2 - 1}{4(2\ell - 1)^2} \right) \right.
\]
\[
+ \left( \frac{1}{4} + (2\ell)^2 \log \frac{4(2\ell)^2 - 1}{4(2\ell)^2} \right)
\]
\[
= \frac{1}{8} + \sum_{\ell=1}^{\infty} \left( - (2\ell - 1)^2 \log \frac{4(2\ell - 1)^2 - 1}{4(2\ell - 1)^2} \right.
\]
\[
+ (2\ell)^2 \log \frac{4(2\ell)^2 - 1}{4(2\ell)^2} \right)
\]
\[
= \frac{1}{8} + \sum_{\ell=1}^{\infty} \log \left( \frac{4\ell - 1}{4\ell} \right)^{4\ell-1} \left( \frac{4\ell + 1}{4\ell} \right)^{4\ell} \left( \frac{4\ell - 2}{4\ell - 3} \right)^{(2\ell-1)^2}
\]
\[
= \frac{1}{8} + \sum_{\ell=1}^{\infty} \log \left( \frac{4\ell - 1}{4\ell - 3} \right)^{2\ell-1} \left( \frac{4\ell + 1}{4\ell} \right)^{2\ell}
\]
\[
+ \sum_{\ell=1}^{\infty} \log \left( \frac{4\ell + 1}{4\ell} \right)^{2\ell^2+2\ell} \left( \frac{4\ell - 2}{4\ell - 3} \right)^{2(2\ell-1)^2}
\]
\[
= \frac{2G}{\pi} - \frac{3}{8} + 2 \sum_{\ell=1}^{\infty} \log \left( \frac{4\ell + 1}{4\ell} \right)^{2\ell^2+\ell} \left( \frac{4\ell - 2}{4\ell - 3} \right)^{(2\ell-1)^2}.
\]
Therefore, by Formula 13, it follows that
\[
\frac{7}{4\pi^2} \zeta(3) = \frac{3}{16} + \log \prod_{\ell=1}^{\infty} \frac{(4\ell)(2\ell)^2(4\ell - 3)(2\ell - 3)}{(4\ell + 1)(2\ell + \ell)(4\ell - 2)^2}.
\]

Now the latter infinite product can be written as
\[
\lim_{N \to \infty} \prod_{\ell=1}^{N} 2^{4\ell - 1} \frac{(2\ell)(2\ell)^2(4(\ell - 1) + 1)}{(2\ell - 1)(2\ell - 1)^2(4\ell + 1)(2\ell + \ell)} = \lim_{N \to \infty} \left( \frac{2}{4N + 1} \prod_{\ell=1}^{N} \frac{(2\ell)}{(2\ell - 1)(2\ell - 1)^2} \right)
\]
\[
= \lim_{N \to \infty} \frac{2^{2N^2 + N}}{(4N + 1)^{2N^2 + N}} \prod_{\ell=1}^{N} \frac{(2\ell)}{(2\ell - 1)} \frac{(2\ell + 1)}{(2\ell + 1)^2} \frac{2}{(2N + 2)^{(2N + 1)(N + 1)}}
\]
\[
\times \prod_{\ell=1}^{N} \frac{(2\ell + 2)}{(2\ell + 1)^{(2\ell + 1)^2}} \frac{(2\ell + 2)}{(2\ell + 1)^{(2\ell + 1)^2}} = \lim_{N \to \infty} \left( e^{\frac{N + 1}{2}} \frac{4N + 2}{4N + 1} \frac{2}{\sqrt{e}} \prod_{\ell=1}^{N} \frac{(2\ell + 2)}{(2\ell + 1)^{(2\ell + 1)^2}} \right).
\]

We claim that
\[
\lim_{N \to \infty} \frac{e^{\frac{N + 1}{2}}}{\left(1 - \frac{1}{4N + 2}\right)^{2(N + 1)N} \left(1 + \frac{1}{2N + 1}\right)^{(2N + 1)(N + 1)}} = e^{-\frac{3}{16}}.
\]

Indeed, by Lemma 16, we have
\[ \lim_{N \to \infty} \frac{e^{N + \frac{1}{4}}}{\left( 1 + \frac{1}{2N + 1} \right)^{(2N + 1)^2}} = 1 = \lim_{N \to \infty} \frac{e^{\frac{1}{4}}}{\left( 1 + \frac{1}{2N + 1} \right)^{(2N + 1)^2}} \]

and

\[ \lim_{N \to \infty} \frac{e^{-\frac{N}{2} - \frac{5}{16}}}{\left( 1 - \frac{1}{4N + 2} \right)^{(4N + 2)^2}} = 1 = \lim_{N \to \infty} \frac{e^{\frac{1}{4}}}{\left( 1 - \frac{1}{4N + 2} \right)^{(4N + 2)^2}} , \]

hence (15) follows. Combining (14) with (15) gives

\[ e^{7\zeta(3)/4\pi^2} = \frac{2}{\sqrt{e}} \prod_{l=1}^{\infty} \frac{(2l)(2l+1)(2l+2)(2l+3)}{(2l+1)^2} . \]

Therefore,

\[ e^{7\zeta(3)/4\pi^2} = \frac{2}{\sqrt{e}} \lim_{m \to \infty} \prod_{n=1}^{m} \frac{(2n)(2n+1)n}{\sqrt{e} (2n+1)^2} \]

\[ = \frac{(2m+2)(2m+1)(m+1)}{\sqrt{e}} \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2m+1)^2} \]

\[ = e^{\frac{1}{4}} \lim_{m \to \infty} \prod_{n=1}^{m+1} \frac{1}{\sqrt{e}} \left( 1 - \frac{1}{n+1} \right)^{\frac{n(n+1)}{2}(-1)^n} . \]

and this completes the proof of formula (1).

It remains to prove formulas (2) and (3). Note that

\[ \prod_{n=1}^{m+1} \sqrt{e} \left( 1 - \frac{1}{n+1} \right)^{\frac{n(n+1)}{2}(-1)^n} = \frac{(1 - \frac{1}{2m+2})^{(2m+1)(m+1)}}{\sqrt{e}^{(m+\frac{1}{2})}} \prod_{n=1}^{m+1} \sqrt{e} \left( 1 - \frac{1}{n+1} \right)^{\frac{n(n+1)}{2}(-1)^n} . \]

Therefore, formula (2) will follow from formula (1) once we show that

\[ \lim_{m \to \infty} \frac{e^{-(2m+\frac{1}{2})}}{(1 - \frac{1}{2m+2})^{(2m+2)(2m+1)}} = e. \]

This follows by writing \((2m+2)(2m+1)\) as \((2m + 2)^2 - (2m + 2)\) and using Lemma 16.

Now multiplying formulas (1) and (2) together and squaring gives

\[ e^{7\zeta(3)/4\pi^2} = \frac{1}{\sqrt{e}} \prod_{n=1}^{2m} \left( \frac{2n+1}{2m+2} \right)^{(2m+1)(2m+2)} \prod_{n=1}^{2m} \left( 1 - \frac{1}{n+1} \right)^{2n(n+1)(-1)^n} \]

\[ = \frac{1}{\sqrt{e}} \left( \frac{(2m+2)^{2m+2}}{(2m+1)^{6m+2}} \right)^{2m+1} \left( 2^{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m)^2} / (1^2 \cdot 3^2 \cdot 5^2 \cdots (2m-1)(2m-1)^2) \right)^{4} \cdot \]
Formula (3) is now a consequence of the equality
\[
\frac{1}{\sqrt{e}} = \lim_{m \to \infty} \left( \frac{2m + 1}{2m + 2} \right)^{m+2}.
\]

### 4 Proof of Facts 3 and 4

By formula (3) and its proof, it suffices to show that the total exponent of \(k+1\) in the infinite product expansion given by Guillera and Sondow [5, Example 5.3] equals \((-1)^{k+1} (k+1)^2\), for all \(k \in \mathbb{N}\). The exponent in question equals

\[
(\frac{-1}{2})^{k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n^2 + n}{2^{n+3}} = (\frac{-1}{2})^{k+1} \sum_{n=k}^{\infty} \frac{1}{8 \cdot (k!)} \binom{n}{2} \frac{n^2 (n-1) \cdots (n-k+1)}{2^n} - 2 \sum_{n=k}^{\infty} \frac{(n+1) n \cdots (n-k+1)}{2^n}
\]

\[
= (\frac{-1}{2})^{k+1} \frac{1}{8 \cdot (k!)} \left( \sum_{m=0}^{\infty} \binom{m+k+1}{m+k} \binom{m+k}{m} \cdots \binom{m+1}{1} \right).
\]

We have the following lemma:

**Lemma 17.** For all \(k \in \mathbb{N}\), we have

\[
\sum_{m=0}^{\infty} \frac{(m+k+1) (m+k) \cdots (m+1)}{2^m} = 2^{(k+2)} \cdot ((k+1)!).
\]

**Proof.** This follows by term-by-term \((k+1)\)-fold differentiation of the geometric series

\[
\sum_{n=0}^{\infty} x^n = (1-x)^{-1}
\]

and subsequent evaluation at \(x = \frac{1}{2}\). □

Therefore, by Lemma 17, the exponent in question equals

\[
\frac{(-1)^{k+1}}{8 \cdot (k!)} \left( 8 \cdot ((k+2))! - 8 \cdot ((k+1)!) \right) = (-1)^{k+1} (k+1)^2,
\]

which completes the proof of Fact 3.

We will now show that, apart from the factor \(e^{-\frac{1}{2}}\) on the left-hand side of formula (4), the product expansion given by the latter formula and the product expansion given by Guillera and Sondow [5, Example 5.5] are equivalent. In other words, we will show that the total
exponent of $2k + 1$ in the infinite product expansion of $e^{\frac{G}{\pi}}$ given by Guillera and Sondow [5, Example 5.5] equals $(-1)^{k+1} \left( k + \frac{1}{2} \right)$, for all $k \in \mathbb{N}$. Since the infinite series involved is only conditionally convergent, the discrepancy involving $e^{-\frac{1}{2}}$ can be explained by means of Riemann’s theorem on rearrangements of conditionally convergent series. The exponent in question equals

\[
(-1)^{k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n}{2^{n+2}} = \frac{(-1)^{k+1}}{4 \cdot (k!)} \sum_{n=k}^{\infty} \frac{n^2 (n-1) \cdots (n-k+1)}{2^n} - \sum_{n=k}^{\infty} \frac{n (n-1) \cdots (n-k+1)}{2^n}
\]

\[
= \frac{(-1)^{k+1}}{4 \cdot (k!)} \left( \frac{1}{2^k} \sum_{m=0}^{\infty} \frac{(m+k+1) (m+k) \cdots (m+1)}{2^m} - \frac{1}{2^k} \sum_{m=0}^{\infty} \frac{(m+k) (m+k-1) \cdots (m+1)}{2^m} \right).
\]

By Lemma 17, this equals

\[
\frac{(-1)^{k+1}}{4 \cdot (k!)} \left( 4 \cdot ((k+1)! \right) - 2 \cdot (k!)) = (-1)^{k+1} \left( k + \frac{1}{2} \right),
\]

as required, and this completes the proof of Fact 4.

5 Concluding remarks

Remark 18. The identities

\[
\sum_{n=k}^{\infty} \binom{n}{k} \frac{n^2 + n}{2^{n+3}} = (k+1)^2,
\]

\[
\sum_{n=k}^{\infty} \binom{n}{k} \frac{n}{2^{n+2}} = k + \frac{1}{2}
\]

which were used in the proofs of Facts 3 and 4 can also be very easily established by the Wilf-Zeilberger method via the use of Zeilberger’s Maple package EKHAD (see [9]).

Remark 19. One way to account for the fact that the products discussed in this paper are so closely tied to the ones studied by Guillera and Sondow in [5] is by noticing that they are related via Euler transformations. For instance, using the latter formula in the previous remark, one has

\[
\lim_{m \to \infty} \sum_{k=1}^{2m} (-1)^k \log \frac{2k-1}{2k+1} = \lim_{m \to \infty} \sum_{k=1}^{2m} (-1)^k \log \frac{2k-1}{2k+1} \sum_{n=k}^{\infty} \binom{n}{k} \frac{n-1}{2^{n+2}}.
\]

If we interchange the summation on the right-hand side (an Euler transformation) the relation between formula (4) and the formula given in Fact 4 becomes evident.
Remark 20. The formulas in Propositions 2 and 1 are reminiscent of some powerful statements that deserve to be more widely known. We refer the reader to Finch’s book [4] for a wealth of information regarding such statements involving classical constants. For instance, the following function (first introduced by Borwein and Dykshoorn in [3]):

\[ D(x) = \lim_{m \to \infty} \prod_{n=1}^{2m+1} \left(1 + \frac{x}{n}\right)^{(-1)^{n+1}} = e^x \lim_{m \to \infty} \prod_{n=1}^{2m} \left(1 + \frac{x}{n}\right)^{(-1)^{n+1}}. \]

Certain values of this function are related to some classical constants. Melzak proved in [7] that \( D(2) = \frac{\pi e^2}{7} \). In [3], Borwein and Dykshoorn generalized Melzak’s result and explicitly determined the values of \( D(x) \) at all rational \( x \) having denominator 1, 2 or 3. Interestingly enough, some of the resulting evaluations involve Catalan’s constant, the Glaisher-Kinkelin constant and \( \Gamma\left(\frac{1}{4}\right) \). We have not been able to show that any of the formulas in Propositions 2 or 1 is a direct consequence of the latter evaluations.

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References


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