Representation of Integers by Near Quadratic Sequences

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Abstract
Following a statement of the well-known Erdős-Turán conjecture, Erdős mentioned the following even stronger conjecture: if the $n$-th term $a_n$ of a sequence $A$ of positive integers is bounded by $\alpha n^2$, for some positive real constant $\alpha$, then the number of representations of $n$ as a sum of two terms from $A$ is an unbounded function of $n$. Here we show that if $a_n$ differs from $\alpha n^2$ (or from a quadratic polynomial with rational coefficients $q(n)$) by at most $o(\sqrt{\log n})$, then the number of representations function is indeed unbounded.

1 Introduction
In 1941, Erdős and Turán [5] conjectured that if a sequence $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ of positive integers is an asymptotic basis of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers,
i.e., if all large enough integers \( n \) are sums of two terms from \( A \), then the number of representations \( r_A(n) = \left| \{(a, a_j) \in A \times A : a_i + a_j = n \} \right| \) of \( n \), as a sum of two terms from \( A \), is unbounded. This is the well-known “Erdős-Turán conjecture”. A few years later (the earliest we are aware of), in 1955 and 1956, Erdős [6], and Erdős and Fuchs [7] asserted that an even stronger conjecture would be that if \( a_n \leq \alpha n^2 \), for all \( n \), with a real constant \( \alpha > 0 \), then \( \limsup r_A(n) = \infty \). This came to be known as the “generalized Erdős-Turán conjecture”. It is indeed stronger than the former one, since if \( A \) is an asymptotic basis of \( \mathbb{N} \), then \( a_n \ll n^2 \) [13, p. 105].

Much work has been done concerning the “Erdős-Turán conjecture”, e.g., [3, 7, 8, 16, 1, 21, 19], including disproofs of analogues of this conjecture in many semigroups other than \( \mathbb{N} \), e.g., [20, 16, 17, 11, 12, 2, 14]. In contrast, much less has been done about the “generalized Erdős-Turán conjecture”. In a previous, co-authored, paper [9], we studied the class of sequences that can replace \( \{\alpha n^2\} \) in the condition \( a_n \leq \alpha n^2 \) for all \( n \), to imply that \( r_A(n) \) is unbounded, and we gave several statements equivalent to the “generalized Erdős-Turán conjecture”. In particular, we showed that if the conjecture holds with \( \alpha = 1 \), then it holds with any \( \alpha > 0 \). Moreover, it is not difficult to see that if \( a_n = o(n^2) \), then the conjecture holds [9, 10]. So we can essentially focus on the case where \( a_n \) is, in a sense, “close” to a constant multiple of \( n^2 \), or to a quadratic polynomial in \( n \). This is basically the goal of the present paper. We thus show that if \( |a_n - \alpha n^2| = o(\sqrt{\log n}) \), with a real constant \( \alpha > 0 \), or if \( |a_n - q(n)| = o(\sqrt{\log n}) \), where \( q(n) \) is a quadratic polynomial with rational coefficients, then the representation function \( r_A(n) \) of \( A \) is unbounded.

2 Technical tools

Let \( C = \{c_1 < c_2 < \cdots < c_n < \cdots \} \subset \mathbb{R}^+ \) be a strictly increasing sequence, in the set \( \mathbb{R}^+ \) of real numbers \( \geq 0 \). For any \( x \in \mathbb{R}^+ \), let \( C[x] = C \cap [0, x] = \{c \in C : c \leq x\} \), and \( C(x) = |C[x]| \) the cardinality of \( C[x] \). Note that \( C(x) \) is finite for every \( x \geq 0 \) if and only if the sequence \( C \) is unbounded. This is in particular true when \( c_{n+1} - c_n \geq 1 \) for large enough \( n \), and more particularly if \( C \) is a subset of the set \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \) of natural numbers.

The subset \( C + C \) is defined by \( C + C = \{c + d : (c, d) \in C \times C\} \).

Now let \( A = \{a_1 < a_2 < \cdots < a_n < \cdots \} \subset \mathbb{N} \) be a strictly increasing sequence of natural numbers. In addition to the above notions, valid for \( A \) as for \( C \), the representation function \( r_A \) of \( A \) is defined by \( r_A(n) = \left| \{(a, b) \in A \times A : a + b = n\} \right| \), for \( n \in \mathbb{N} \), and we set \( s(A) = \sup_{n \in \mathbb{N}} r_A(n) \), in \( \mathbb{N} = \mathbb{N} \cup \{\infty\} \).

In the sequel, \( i, j, k, l, m, n \) generally denote positive integers, unless it is specified that they lie in \( \mathbb{N} \), i.e., that they are integers \( \geq 0 \), while \( x, y \) denote real numbers \( \geq 0 \), i.e., they lie in \( \mathbb{R}^+ \).

Note that if \( A = \{a_1 < a_2 < \cdots < a_n < \cdots \} \subset \mathbb{N}^* \), where \( \mathbb{N}^* = \{1, 2, 3, \ldots \} \) is the set of positive integers, then \( a_n \geq n \) for all \( n \in \mathbb{N}^* \).

For any \( x \in \mathbb{R}^+ \), let

\[
U_A(x) = \left| \{(a, b) \in A \times A : a + b \leq x\} \right| = \sum_{0 \leq n \leq x} r_A(n). \tag{1}
\]
Then
\[ U_A(x) = \sum_{n \in \{A-x\}} r_A(n) \leq \sum_{n \in \{A+x\}} s(A) = (A + A)(x) \cdot s(A) \] (2)

and
\[
A(x)^2 = |\{(a, b) \in A \times A : a, b \leq x\}| \leq |\{(a, b) \in A \times A : a + b \leq 2x\}| = U_A(2x) \leq (A + A)(2x) \cdot s(A),
\] (3)

so that, for all \(x \in \mathbb{R}^+\),
\[
\frac{(A + A)(2x)}{A(x)^2} \cdot s(A) \geq 1.
\] (4)

Define
\[
h(A) = \liminf_{x \to \infty} \frac{(A + A)(2x)}{A(x)^2}.
\] (5)

**Lemma 1.** If \(h(A) = 0\), then \(s(A) = \infty\).

**Proof.** This follows immediately from (4). \(\square\)

**Corollary 2.** If \(\liminf_{n \to \infty} \frac{A(x)}{\sqrt{x}} > 0\) and \(\liminf_{n \to \infty} \frac{(A + A)(x)}{x} = 0\), then \(h(A) = 0\), and therefore \(s(A) = \infty\).

**Proof.** By assumption, \(\limsup_{n \to \infty} \frac{\sqrt{x}}{A(x)} = \frac{1}{\liminf_{n \to \infty} \frac{A(x)}{\sqrt{x}}}\) is finite, while \(\liminf_{n \to \infty} \frac{(A + A)(2x)}{2x} = 0\). So, using properties of the lower and upper limits, we get
\[
h(A) = \liminf_{x \to \infty} \frac{(A + A)(2x)}{A(x)^2} = 2 \liminf_{x \to \infty} \frac{(A + A)(2x)}{2x} \left(\frac{\sqrt{x}}{A(x)}\right)^2 \leq 2 \left(\liminf_{x \to \infty} \frac{(A + A)(2x)}{2x}\right) \cdot \left(\limsup_{x \to \infty} \frac{\sqrt{x}}{A(x)}\right)^2 = 0.
\]

The conclusion follows from Lemma 2.1. \(\square\)

**Lemma 3.** Let \(A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq \mathbb{N}^+\) be a strictly increasing sequence of positive integers, and \(C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subseteq \mathbb{R}^+\). For \(x \in \mathbb{R}^+\), set \(e(x) = \sup_{n \leq x} |a_n - c_n|\). We then have, for all \(x \in \mathbb{R}^+\),
\[
(A + A)(x) \leq (4e(x) + 1) \cdot (C + C)(x + 2e(x)).
\] (6)

If we further assume that \(c_1 \geq 1\) and \(c_{n+1} - c_n \geq 1\) for all \(n \geq 1\), we then also have, for all \(x \in \mathbb{R}^+\),
\[
A(x) \geq C(x - e(x)).
\] (7)
**Proof.** Note first that the function \( e(x) \) is increasing, in the sense that \( x \leq y \) implies \( e(x) \leq e(y) \).

Note also that, since \( A \subset N^* \), we have \( i \leq a_i \) for all \( i \). So, for \( n \leq x \), if \( n = a_i + a_j \), then \( i \leq a_i \leq n \leq x \) and similarly \( j \leq x \), and therefore \( |n - c_i - c_j| = |a_i + a_j - c_i - c_j| \leq |a_i - c_i| + |a_j - c_j| \leq 2e(x) \). Hence

\[
(A + A)[x] = \{n \leq x : \exists i, j, n = a_i + a_j\} \subset \{n \leq x : \exists i, j, |n - c_i - c_j| \leq 2e(x)\},
\]

and setting \( s = c_i + c_j \), we get \( s \in C + C \) and \( |n - s| \leq 2e(x) \), so that \( s \leq n + 2e(x) \leq x + 2e(x) \), and therefore

\[
\{n \leq x : \exists i, j, |n - c_i - c_j| \leq 2e(x)\} \subset \{n : \exists s \in (C + C)[x + 2e(x)], |n - s| \leq 2e(x)\}.
\]

Thus

\[
(A + A)[x] \subset \bigcup_{s \in (C + C)[x + 2e(x)]} ([s - 2e(x), s + 2e(x)] \cap \mathbb{N}),
\]

and therefore

\[
(A + A)(x) \leq \sum_{n \in (C + C)[x + 2e(x)]} (4e(x) + 1) = (C + C)(x + 2e(x)) \cdot (4e(x) + 1).
\]

This proves (6).

Now, if \( c_1 \geq 1 \) and \( c_{n+1} - c_n \geq 1 \) for all \( n \), then \( c_n \geq n \) for all \( n \). So if \( c_n \leq x - e(x) \), then \( n \leq c_n \leq x \), so that \( |a_n - c_n| \leq e(x) \), and therefore \( a_n \leq c_n + e(x) \leq x \).

Hence \( \{n : c_n \leq x - e(x)\} \subset \{n : a_n \leq x\} \), and thus

\[
C(x - e(x)) = |\{n : c_n \leq x - e(x)\}| \leq |\{n : a_n \leq x\}| = A(x),
\]

which proves (7). \( \square \)

**Lemma 4.** Let \( A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset N^* \) and \( C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subset \mathbb{R}^+ \) be two strictly increasing sequences in \( N^* \) and in \( \mathbb{R}^+ \), respectively. For \( x \in \mathbb{R}^+ \), set \( e(x) = \sup_{n \leq x} |a_n - c_n| \). Assume that \( e(x) \) is not identically zero, and that \( c_1 \geq 1 \) and \( c_{n+1} - c_n \geq 1 \) for all \( n \geq 1 \). Then the condition

\[
\liminf_{x \to \infty} \frac{e(2x) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2} = 0 \tag{H}
\]

implies that \( h(A) = 0 \), and therefore \( s(A) = \infty \).

**Proof.** Since \( e(x) \) is increasing and not identically zero, there exists a real constant \( t > 0 \) such that \( e(x) \geq \frac{1}{t} \) for large enough \( x \). In view of the inequalities (6) and (7) in Lemma 2.3, we have

\[
\frac{(A + A)(2x)}{A(x)^2} \leq \frac{(4e(2x) + 1) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2}.
\]
Moreover, for large enough $x$, we have $t\cdot e(2x) \geq 1$, and therefore $4e(2x)+1 \leq (4+t)\cdot e(2x)$. Thus

$$\frac{(A+A)(2x)}{A(x)^2} \leq (4+t)\frac{e(2x)\cdot(C+C)(2x+2e(2x))}{C(x-e(x))^2},$$

for large enough $x$, so that the condition (H) implies that $\lim \inf_{x \to \infty} \frac{(A+A)(2x)}{A(x)^2} = 0$, i.e., $h(A) = 0$, and therefore, by Lemma 2.1, $s(A) = \infty$. \hfill \Box

Remark 5. The scope of Lemma 2.4 is broader than it seems to be. Indeed, for a subset $A$ of $\mathbb{N}$, modifying, removing or adding finitely many elements does not modify the fact that $s(A)$ is infinite or finite. Thus Lemma 2.4 can be used in more general situations than specified by its assumptions, as shown by the next result.

**Fundamental Lemma 6.** Let $B = \{b_1 < b_2 < \cdots < b_n < \cdots\} \subset \mathbb{N}$ and $D = \{d_1 < d_2 < \cdots < d_n < \cdots\} \subset \mathbb{R}^+$ be two strictly increasing sequences in $\mathbb{N}$ and in $\mathbb{R}^+$ respectively. Assume that there exists an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a positive integer $m$ such that $d_m \geq 1$, $d_{n+1} - d_n \geq 1$ for $n \geq m$, and $\sup_{m \leq n \leq x}|b_n - d_n| \leq f(x)$ for $x \geq m$. Then the condition

$$\lim \inf_{x \to \infty} \frac{f(2x)\cdot(D+D)(2x+2f(2x))}{D(x-f(x))^2} = 0$$

(K)

implies that $s(B) = \infty$.

**Proof.** For $n \in \mathbb{N}^*$, set $a_n = b_{n+m}$ and $c_n = d_{n+m}$, and let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}^*$ and $C = \{c_1 < c_2 < \cdots < c_n < \cdots\} \subset \mathbb{R}^+$ be the strictly increasing sequences, in $\mathbb{N}^*$ and $\mathbb{R}^+$, obtained by deleting the first $m$ terms of $B$ and $D$ respectively. Then $c_1 = d_{m+1} \geq 2$ and $c_{n+1} - c_n = d_{n+m+1} - d_{n+m} \geq 1$ for $n \geq 1$. Moreover, setting $e(x) = \sup_{n \leq x}|a_n - c_n|$, for $x \in \mathbb{R}^+$, and using the assumptions on $B$ and $D$, we have

$$e(x) = \sup_{n \leq x}|a_n - c_n| = \sup_{n \leq x}|b_{n+m} - d_{n+m}| = \sup_{m \leq i \leq x+m}|b_i - d_i| \leq f(x+m).$$

Thus, setting $y = x+m$, we have $e(x) \leq f(y)$, and since the functions $e$ and $f$ are increasing,

$$e(2x) \leq f(2x+m) \leq f(2y).$$

Also, taking into account that $C \subset D$ and $C+C \subset D+D$, so that $(C+C)(t) \leq (D+D)(t)$ for all $t \in \mathbb{R}^+$, and that the function $t \mapsto (C+C)(t)$ is increasing, we get

$$(C+C)(2x+2e(2x)) \leq (C+C)(2y+2f(2y)) \leq (D+D)(2y+2f(2y)).$$

Thus

$$e(2x)\cdot(C+C)(2x+2e(2x)) \leq f(2y)\cdot(D+D)(2y+2f(2y)), \quad (8)$$

for $x \in \mathbb{R}^+$, and $y = x+m$. 


Moreover, for $t \geq m$, we have
\[
D(t) - C(t) = |\{d_n \in D : d_n \leq t\}| - |\{c_n \in C : c_n = d_{n+m} \leq t\}| = m
\]
and
\[
C(t) - C(t-m) = |\{c_n \in C : t-m < c_n \leq t\}| \leq m,
\]
since $c_{n+1} - c_n \geq 1$ for all $n \in \mathbb{N}^*$, so that $C(t) \leq C(t-m) + m$ and $D(t) = C(t) + m \leq C(t-m) + 2m$. Therefore $C(t-m) \geq D(t) - 2m$ for $t \geq m$. Hence, taking into account that the function $t \mapsto C(t)$ is increasing and that $e(x) \leq f(y)$ we get, for large enough $x$,
\[
C(x - e(x)) \geq C(x - f(y)) = C(y - m - f(y)) \geq D(y - f(y)) - 2m. \tag{9}
\]
It follows from (8) and (9) that, for large enough $x$ and for $y = x + m$,
\[
\frac{e(2x) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2} \leq \frac{f(2y) \cdot (D + D)(2y + 2f(2y))}{(D(y - f(y)) - 2m)^2}. \tag{10}
\]
Set $P(x) = f(2x) \cdot (D + D)(2x + 2f(2x))$ and $Q(x) = D(x - f(x))$, and suppose that the condition (K) is satisfied, i.e., that $\liminf_{x \to \infty} \frac{P(x)}{Q(x)^2} = 0$. Then there exists a strictly increasing sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^+$, tending to infinity, such that $\lim_{n \to \infty} \frac{P(x_n)}{Q(x_n)^2} = 0$. Since $P(x)$ is an increasing unbounded function, $\lim_{n \to \infty} P(x_n) = \infty$, and therefore the sequence $(Q(x_n))_{n \geq 1}$ is unbounded. So there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}^*}$ of $(x_n)_{n \geq 1}$ such that $\lim_{k \to \infty} Q(x_{n_k}) = \infty$,

while $\lim_{k \to \infty} \frac{P(x_{n_k})}{Q(x_{n_k})^2} = 0$. Hence $\lim_{k \to \infty} \frac{P(x_{n_k})}{(Q(x_{n_k}) - 2m)^2} = 0$, and therefore
\[
\liminf_{y \to \infty} \frac{f(2y) \cdot (D + D)(2y + 2f(2y))}{(D(y - f(y)) - 2m)^2} = \liminf_{x \to \infty} \frac{P(x)}{(Q(x) - 2m)^2} = 0.
\]
It then follows from (10) that $\liminf_{x \to \infty} \frac{e(2x) \cdot (C + C)(2x + 2e(2x))}{C(x - e(x))^2} = 0$. Thus the condition (H) of Lemma 2.4 holds, and therefore, in view of this Lemma, $s(A) = \infty$. As $A \subset B$, it follows that $s(B) = \infty$ too.

**Remark 7.** In the statement of Lemma 2.6, we may replace $D$ by $D' = D + \gamma$, i.e., $d_n$ by $d'_n = d_n + \gamma$ ($n \in \mathbb{N}^*$), where $\gamma$ is any fixed real number, since a translation of the general term of $D$ does not affect the condition (K).

### 3 Main results

**Theorem 8.** Let $A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subset \mathbb{N}$ be a strictly increasing sequence of natural numbers, and $q(x) = ax^2$ with a real number $a > 0$. If the function $e(x) = \sup_{n \leq x} |a_n - q(n)|$ ($x \in \mathbb{R}^+$) satisfies $e(x) = o\left(\sqrt{\log x}\right)$ as $x \to \infty$, then $s(A) = \infty$. 
Proof. We apply Lemma 2.6 to \( B = A \) and \( D = \{ q(1) < q(2) < \cdots < q(n) < \cdots \} \). Indeed, the sequence \( \langle q(n) \rangle_{n \geq 1} \) is strictly increasing and unbounded, with \( q(n+1) - q(n) = \alpha (2n+1) \) unbounded too, so that \( q(n) \geq 1 \) and \( q(n+1) - q(n) \geq 1 \) for large enough \( n \). There remains to show that the condition (K) holds for \( f(x) = e(x) \).

Let \( S = \{ n^2 : n \in \mathbb{N}^* \} \). By a classical result of Landau [15], there exists a constant \( c > 0 \) such that \((S + S) (x) \sim c \frac{x}{\log x} \) as \( x \to \infty \).

For \( m, n \in \mathbb{N}^* \) and \( x \in \mathbb{R}^+ \), as \( q(m) + q(n) \leq x \) is equivalent to \( m^2 + n^2 \leq \frac{x}{\alpha} \), we have
\[
(D + D) (x) = (S + S) \left( \frac{x}{\alpha} \right) \sim c \frac{x}{\alpha \sqrt{\log x}},
\]
so that
\[
(D + D) (x) \leq c_1 \frac{x}{\sqrt{\log x}},
\]
for large enough \( x \), with a constant \( c_1 > \frac{c}{\alpha} \).

Moreover, as \( q(n) \leq x \) if and only if \( n \leq \sqrt{\frac{x}{\alpha}} \), we also have \( D(x) = \left[ \sqrt{\frac{x}{\alpha}} \right] > \sqrt{\frac{x}{\alpha}} - 1 \).

It follows that, for large enough \( x \),
\[
\frac{e(2x) \cdot (D + D) (2x + 2e(2x))}{D(x - e(x))^2} \leq \frac{c_1 \cdot e(2x) \cdot (2x + 2e(2x))}{\sqrt{\log (2x + 2e(2x))} \left( \sqrt{\frac{x-e(x)}{\alpha}} - 1 \right)^2} = \frac{c_1 \cdot e(2x) \cdot (2x + 2e(2x))}{\sqrt{\log (2x + 2e(2x))} \left( \sqrt{x - e(x) - \sqrt{\alpha}} \right)^2}.
\]

As \( e(x) = o \left( \sqrt{\log x} \right) \),
\[
\frac{e(2x) \cdot (2x + 2e(2x))}{\sqrt{\log (2x + 2e(2x))} \left( \sqrt{x - e(x) - \sqrt{\alpha}} \right)^2} \sim \frac{2x \cdot e(2x)}{\sqrt{\log (2x)} \cdot x} \sim \frac{2e(2x)}{\sqrt{\log (2x)}},
\]
and, since \( e(x) = o \left( \sqrt{\log x} \right) \), we have \( \lim_{x \to \infty} \frac{2e(2x)}{\sqrt{\log (2x)}} = 0 \). Therefore
\[
\lim_{x \to \infty} \frac{e(2x) \cdot (D + D) (2x + 2e(2x))}{D(x - e(x))^2} = 0,
\]
and the condition (K) holds. Thus, by Lemma 2.6, \( s(B) = \infty \), i.e., \( s(A) = \infty \).

Remark 9. In the statement of Theorem 3.1, we may replace \( q(x) = \alpha x^2 \) by \( q(x) = \alpha x^2 + \gamma \), where \( \gamma \) is any real constant, in view of Remark 2.7.

Also, if \( A = \{ a_n = [\alpha n^2 + \gamma] : n \in \mathbb{N} \} \) is the set of the integral parts \( [\alpha n^2 + \gamma] = [q(n)] \), then \( s(A) = \infty \), since \( e(x) = \sup_{n \leq x} |a_n - q(n)| \leq 1 \) trivially satisfies the condition in Theorem 3.1.
Theorem 10. Let \( A = \{a_1 < a_2 < \cdots < a_n < \cdots \} \subset \mathbb{N} \) and \( q(x) \) be a quadratic polynomial with rational coefficients and positive leading coefficient. If the function \( e(x) = \sup_{n \leq x} |a_n - q(n)| \) ( \( x \in \mathbb{R}^+ \)) satisfies \( e(x) = o \left( \sqrt[4]{\log x} \right) \) as \( x \to \infty \), then \( s(A) = \infty \).

Proof. As \( q(x) \) has rational coefficients, there exist integers \( a, b, c, d \), with \( a, d > 0 \), such that \( dq(x) = (ax + b)^2 + c \).

Let \( b_n = da_n - c \) and \( d_n = (an + b)^2 \), for \( n \in \mathbb{N}^* \). Clearly, there exists \( m \in \mathbb{N}^* \) such that \( b_m \geq 1 \), \( d_m \geq 1 \) and \( d_{m+1} - d_n \geq 1 \) for \( n \geq m \). Set \( B = \{b_n : n \geq m\} \) and \( D = \{d_n : n \geq m\} \). Then \( B \) and \( D \) are strictly increasing sequences in \( \mathbb{N} \), and, for all \( n \geq m \),

\[
|d_n - b_n| = |(an + b)^2 - da_n + c| = d|q(n) - a_n|.
\]

For \( x > m \), let \( f(x) = \sup_{m \leq n \leq x} |d_n - b_n| \), for \( x \in \mathbb{R}^+ \). Then \( f(x) \) is an increasing nonnegative function satisfying \( f(x) \leq d \cdot e(x) \), so that \( f(x) = o \left( \sqrt[4]{\log x} \right) \) (like \( e(x) \)). Thus, we may apply Lemma 2.6, provided we show that the condition (K) is satisfied.

Let \( S = \{n^2 : n \in \mathbb{N}\} \). Then \( D \subset S \), and therefore \( D + D \subset S + S \), so that \( (D + D)(x) \leq (S + S)(x) \), for \( x \in \mathbb{R}^+ \).

By Landau’s theorem \([15] \quad (S + S)(x) \sim c_0 \frac{x}{\sqrt[4]{\log x}} \), with a constant \( c_0 > 0 \). So there exists a constant \( c_1 > 0 \) such that \( (D + D)(x) \leq (S + S)(x) \leq c_1 \frac{x}{\sqrt[4]{\log x}} \), and therefore

\[
(D + D)(2x + 2f(2x)) \leq c_1 \frac{2x + 2f(2x)}{\sqrt[4]{(2x + 2f(2x))}}.
\]  

(11)

Moreover, for \( x > \text{max}(m, b^2) \), if \( n \leq \frac{\sqrt{x}}{a} - |b| \), then \( d_n = (an + b)^2 \leq x \). Hence, for large enough \( x \),

\[
D(x) = |\{n \geq m : d_n \leq x\}| \geq \left| \left\{ n \geq m : n \leq \frac{\sqrt{x}}{a} - \frac{|b|}{a} \right\} \right|
\]

\[
\geq \frac{\sqrt{x} - |b|}{a} - m \geq c_2\sqrt{x} - c_3,
\]

with constants \( c_2, c_3 > 0 \), and therefore

\[
D(x) - f(x) \geq c_2\sqrt{x} - f(x) - c_3.
\]  

(12)

It follows from (11) and (12) that, for large enough \( x \),

\[
\frac{f(2x) \cdot (D + D)(2x + 2f(2x))}{D(x - f(x))^2} \leq c_1 \frac{f(2x) \cdot (2x + 2f(2x))}{\sqrt[4]{(2x + 2f(2x))} \left( c_2\sqrt{x - f(x)} - c_3 \right)^2},
\]

and, since \( f(x) = o \left( \sqrt[4]{\log x} \right) \), we have

\[
\frac{f(2x) \cdot (2x + 2f(2x))}{\sqrt[4]{(2x + 2f(2x))} \left( c_2\sqrt{x - f(x)} - c_3 \right)^2} \sim \frac{2f(2x)}{c_2^2\sqrt[4]{\log x}} = o(1).
\]
Therefore
\[ \liminf_{x \to \infty} \frac{f(2x) \cdot (D + D)(2x + 2f(2x))}{D(x - f(x))^2} = 0. \]
Thus the condition (K) is satisfied, and by Lemma 2.6, \( s(B) = \infty \). As \( B \) is a translate of a homothetic of a subsequence \( A_m = \{a_n : n \geq m\} \) of \( A \), namely \( B = d \cdot A_m + |c| \), we conclude, e.g., see [9], that \( s(A_m) = s(B) = \infty \), and therefore \( s(A) = \infty \).

\[ \square \]

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References


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