



AN INTERESTING DOUBLE INEQUALITY FOR EULER'S GAMMA FUNCTION

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ABSTRACT. In this short paper we derive new and interesting upper and lower bounds for the Euler's gamma function in terms of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$.

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1. INTRODUCTION

It is well known that the Euler's gamma function $\Gamma(z)$ and the psi or digamma function, the logarithmic derivative of the gamma function, are defined as

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du, \quad \operatorname{Re} z > 0$$

and

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0,$$

respectively. The derivatives $\psi', \psi'', \psi''', \dots$ are known as polygamma functions.

The gamma function has been investigated intensively by many authors even recent years. In particular, many authors have published numerous interesting inequalities for this important function (see [2] – [10]). In this note we present new and interesting upper and lower bounds for this function. Throughout we denote by $c = 1.461632144968362\dots$ the only positive zero of the ψ -function (see [1, p. 259, 6.3.19]).

2. MAIN RESULT

The following theorem is our main result.

Theorem 2.1. For all $x \geq c$, the following inequality for the gamma function holds.

$$(2.1) \quad \Gamma(c) \exp[\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1] \leq \Gamma(x) \\ \leq \Gamma(c) \exp [k (\psi(x) e^{\psi(x)} - e^{\psi(x)} + 1)],$$

where γ is Euler-Mascheroni constant, $\Gamma(c) = 0.885603194410889 \dots$; see [1, p. 259; 6.3.9] and $k = 6e^\gamma/\pi^2 = 1.0827621932609 \dots$

Proof. Applying the mean value theorem to the function $\log \Gamma(x)$ on $[u, u + 1]$ with $u > 0$ and using the well known difference equation $\Gamma(u + 1) = u \Gamma(u)$ for the gamma function, there exists a θ depending on u such that for all $u > 0$, $0 < \theta = \theta(u) < 1$ and

$$(2.2) \quad \psi(u + \theta(u)) = \log u.$$

First, we show that the function θ is strictly increasing and θ' is strictly decreasing on $(0, \infty)$. For this purpose put $u = e^{\psi(t)}$ with $t > 0$ in (2.2) to obtain

$$\psi(e^{\psi(t)} + \theta(e^{\psi(t)})) = \psi(t).$$

Since the mapping $t \rightarrow \psi(t)$ from $(0, \infty)$ to $(-\infty, \infty)$ is bijective, we find that

$$(2.3) \quad \theta(e^{\psi(t)}) = t - e^{\psi(t)}, \quad t > 0.$$

Differentiating both sides of this equation, we get

$$(2.4) \quad \theta'(e^{\psi(t)}) = \frac{1}{\psi'(t) e^{\psi(t)}} - 1.$$

By [2, (4.34)] we have

$$(2.5) \quad \psi'(x)e^{\psi(x)} < 1$$

for $x > 0$, so that we conclude $\theta'(e^{\psi(t)}) > 0$, $t > 0$. But since the mapping $t \rightarrow e^{\psi(t)}$ from $(0, \infty)$ to $(0, \infty)$ is also bijective this implies that $\theta'(t) > 0$ for all $t > 0$. Now differentiate both sides of (2.4) to obtain

$$\theta''(e^{\psi(t)}) = -\frac{e^{-2\psi(t)}}{\psi'(t)^3} [(\psi'(t))^2 + \psi''(t)].$$

In [2, (4.39)] it is proved that $[(\psi'(t))^2 + \psi''(t)] > 0$. Using this inequality we have $\theta''(e^{\psi(t)}) < 0$ for $t > 0$. Proceeding as above we conclude that $\theta''(t) < 0$ for all $t > 0$.

To prove the theorem integrate both sides of (2.2) over $1 \leq u \leq x$ to obtain

$$\int_1^x \psi(u + \theta(u)) du = \int_1^x \log u du.$$

Making the change of variable $u = e^{\psi(t)}$ on the left hand side this becomes by (2.3)

$$(2.6) \quad \int_c^{x+\theta(x)} \psi(t)\psi'(t)e^{\psi(t)} dt = x \log x - x + 1.$$

Since $\psi(t) \geq 0$ for all $t \geq c$, and $\psi'(t)e^{\psi(t)} < 1$ by (2.5), this gives for $x \geq 1$

$$x \log x - x + 1 + \log \Gamma(c) \leq \log \Gamma(x + \theta(x)).$$

Replace x by $e^{\psi(x)}$ and then employ (2.3) to get, for $x \geq c$

$$\log \Gamma(c) + [\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1] \leq \log \Gamma(x),$$

which implies the left-hand inequality of (2.1). Since θ' is decreasing we conclude from (2.4) that

$$\theta'(e^{\psi(t)}) = \frac{1}{\psi'(t)e^{\psi(t)}} - 1 \leq \theta'(e^{\psi(1)}) = \frac{6e^\gamma}{\pi^2} - 1$$

for all $t \geq 1$. This implies that $\psi'(t)e^{\psi(t)} > e^{-\gamma}\pi^2/6$ for $t \geq 1$. Hence, using (2.6) we arrive at, after brief simplification,

$$\log \Gamma(x + \theta(x)) \leq \frac{6e^\gamma}{\pi^2} [x \log x - x + 1] + \log \Gamma(c).$$

Now replace x by $e^{\psi(x)}$ and then use (2.3) to get the right inequality in (2.1). This completes the proof of the theorem. \square

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