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AN ASYMPTOTIC EXPANSION

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ABSTRACT. In this paper we study the asymptotic behaviour of the sequence $(r_n)_n$ of the powers of primes. Calculations also yield the evaluation $\sqrt{r_n} - p_n = o\left(\frac{n}{log^s n}\right)$ for every positive integer s, p_n denoting the n-th prime.

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1. INTRODUCTION

One denotes by:

- p_n the n-th prime
- r_n the n-th number (in increasing order) which can be written as a power p^m, $m \geq 2$, of a prime p.
- $\pi(x)$ the number of prime numbers not exceeding x.
- $\tilde{\pi}(x)$ the number of prime powers p^m, $m \geq 2$, not exceeding x.

The asymptotic equivalences

$$\pi(x) \sim \frac{x}{\log x} \tag{1.1}$$

and

$$p_n \sim n \log n \tag{1.2}$$

are well known.

M. Cipolla [1] proves the relations

$$p_n = n(\log n + \log\log n - 1) + o(n) \tag{1.3}$$

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and

$$p_n = n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right) + o\left(\frac{n}{\log n}\right) \tag{1.4}$$

that he generalizes to

Theorem 1.1. *There exists a sequence $(P_m)_{m\geq 1}$ of polynomials with integer coefficients such that, for any integer m,*

$$p_n = n\left[\log n + \log\log n - 1 + \sum_{j=1}^{m} \frac{(-1)^{j-1}P_j(\log\log n)}{\log^j n} + o\left(\frac{1}{\log^m n}\right)\right]. \tag{1.5}$$

In the same paper, M. Cipolla gives recurrence formulae for P_m; he finds that every P_m has degree m and leading coefficient $(m-1)!$.

As far as $(r_n)_n$ is concerned, L. Panaitopol [2] proves the asymptotic equivalence

$$r_n \sim n^2 \log^2 n. \tag{1.6}$$

We prove in this paper that $(r_n)_n$ has an asymptotic expansion comparable to that of Theorem 1.1 .

We will need the next results of L. Panaitopol:

$$\tilde{\pi}(x) - \pi(\sqrt{x}) = O(\sqrt[3]{x}), \tag{1.7}$$

(from [2]), and

Proposition 1.2. *There exist a sequence of positive integers $k_1, k_2, \ldots$ and for every $n \geq 1$ a function α_n, $\lim_{x\to\infty} \alpha_n(x) = 0$, such that:*

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{log^2 x} - \cdots - \frac{k_n(1+\alpha_n(x))}{\log^n x}}. \tag{1.8}$$

Moreover, $k_1, k_2, \ldots$ are given by the recurrence relation

$$k_n + 1! \cdot k_{n-1} + 2! \cdot k_{n-2} + \cdots + (n-1)! \cdot k_1 = n \cdot n!, \quad n \geq 1. \tag{1.9}$$

(from [3]).

We will also establish a result similar to Proposition 1.2 for $\tilde{\pi}(x)$ and the evaluation

$$\sqrt{r_n} - p_n = o\left(\frac{n}{\log^s n}\right)$$

for every positive integer s.

2. On the Asymptotic Behaviour of $\tilde{\pi}$

Proposition 2.1. *For every positive integer n, there exists a function β_n, $\lim_{x\to\infty} \beta_n(x) = 0$, such that*

$$\tilde{\pi}(x) = \frac{\sqrt{x}}{\log\sqrt{x} - 1 - \frac{k_1}{\log\sqrt{x}} - \ldots - \frac{k_{n-1}}{\log^{n-1}\sqrt{x}} - \frac{k_n(1+\beta_n(x))}{\log^n\sqrt{x}}}, \tag{2.1}$$

$(k_n)_n$ being the sequence of (1.9).

Proof. Let us set

$$\tilde{\pi}(x) = \frac{\sqrt{x}}{\log\sqrt{x} - 1 - \frac{k_1}{\log\sqrt{x}} - \ldots - \frac{k_{n-1}}{\log^{n-1}\sqrt{x}} - \frac{k_n(1+\beta_n(x))}{\log^n\sqrt{x}}}. \tag{2.2}$$

(1.8) and (1.7) give us:

$$\sqrt{x}\cdot\frac{k_n[\beta_n(x)-\alpha_n(\sqrt{x})]}{\log^{n+2}x}=O(\sqrt[3]{x}), \tag{2.3}$$

so

$$k_n[\beta_n(x)-\alpha_n(\sqrt{x})]=O\left(\frac{\log^{n+2}x}{\sqrt[6]{x}}\right), \tag{2.4}$$

leading to $lim_{x\to\infty}\beta_n(x)=0.$ □

3. INITIAL ESTIMATES FOR r_n

Equation (2.1) gives:

$$\tilde{\pi}(x)\sim\frac{2\sqrt{x}}{\log x}. \tag{3.1}$$

If we put $x=r_n$, we get

$$n\sim\frac{2\sqrt{r_n}}{\log r_n}, \tag{3.2}$$

so

$$\lim_{n\to\infty}(\log 2+\log\sqrt{r_n}-\log n-\log\log r_n)=0, \tag{3.3}$$

whence

$$\lim_{n\to\infty}\frac{\log\sqrt{r_n}}{\log n}=1, \tag{3.4}$$

leading to:

$$\lim_{n\to\infty}(\log\log r_n-\log 2-\log\log n)=0. \tag{3.5}$$

(3.3) and (3.5) give:

$$\log\sqrt{r_n}=\log n+\log\log n+o(1). \tag{3.6}$$

(2.1) implies

$$\tilde{\pi}(x)=\frac{\sqrt{x}}{\log\sqrt{x}-1+o(1)}. \tag{3.7}$$

For $x=r_n$ we get (in view of (3.6)):

$$\frac{\sqrt{r_n}}{n}=\log n+\log\log n-1+o(1). \tag{3.8}$$

By taking logarithms of both sides we get:

$$\log\sqrt{r_n}-\log n=\log\log n+\log\left[1+\frac{\log\log n-1}{\log n}+o\left(\frac{1}{\log n}\right)\right]. \tag{3.9}$$

For big enough n we have $\left|\frac{\log\log n-1}{\log n}+o\left(\frac{1}{\log n}\right)\right|<1$, which means that we can expand the logarithm. We derive:

$$\log\sqrt{r_n}=\log n+\log\log n+\frac{\log\log n-1}{\log n}+o\left(\frac{1}{\log n}\right). \tag{3.10}$$

(2.1) also gives:

$$\tilde{\pi}(x) = \frac{\sqrt{x}}{\log\sqrt{x} - 1 - \frac{1}{\log\sqrt{x}} + o\left(\frac{1}{\log x}\right)}. \tag{3.11}$$

For $x = r_n$ and in view of (3.4), we obtain:

$$\frac{\sqrt{r_n}}{n} = \log\sqrt{r_n} - 1 - \frac{1}{\log\sqrt{r_n}} + o\left(\frac{1}{\log n}\right). \tag{3.12}$$

(3.10) and (3.12) allow us to write

$$\frac{\sqrt{r_n}}{n} = \log n + \log\log n - 1 + \frac{\log\log n - 1}{\log n} - \frac{1}{\log n\left[1 + \frac{\log\log n}{\log n} + \frac{\log\log n - 1}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right)\right]} + o\left(\frac{1}{\log n}\right). \tag{3.13}$$

For big enough n we have

$$\left|\frac{\log\log n}{\log n} + \frac{\log\log n - 1}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right)\right| < 1;$$

we can therefore use the expansion of $\frac{1}{1+x}$ in (3.13) and we get

$$\sqrt{r_n} = n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right) + o\left(\frac{n}{\log n}\right). \tag{3.14}$$

4. MAIN RESULT

Theorem 4.1. *For every positive integer s we have*

$$\frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^s n}\right). \tag{4.1}$$

Proof. Induction with respect to s.
For $s = 1$ the statement is true because of (1.4) and (3.14).

Now let $s \geq 1$; suppose that

$$\frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^s n}\right). \tag{4.2}$$

(4.2) and (1.5) lead to

$$\sqrt{r_n} = n\left[\log n + \log\log n - 1 + \sum_{j=1}^{s}\frac{(-1)^{j-1}P_j(\log\log n)}{\log^j n} + o\left(\frac{1}{\log^s n}\right)\right]. \tag{4.3}$$

By taking logarithms of both sides in (1.5) we derive

$$\log p_n = \log n + \log\log n + \log\left[1 + \frac{\log\log n - 1}{\log n} + \sum_{j=1}^{s}\frac{(-1)^{j-1}P_j(\log\log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{s+1} n}\right)\right]. \tag{4.4}$$

(1.8) gives us

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_{s+1}}{\log^{s+1} x} + o\left(\frac{1}{\log^{s+1} x}\right)}. \tag{4.5}$$

For $x = p_n$, this relation becomes (in view of (1.2)):

$$\frac{p_n}{n} = \log p_n - 1 - \frac{k_1}{\log p_n} - \dots - \frac{k_{s+1}}{\log^{s+1} p_n} + o\left(\frac{1}{\log^{s+1} n}\right). \tag{4.6}$$

By taking logarithms of both sides in (4.3) we get

$$\log \sqrt{r_n} = \log n + \log\log n + \log\left[1 + \frac{\log\log n - 1}{\log n} + \sum_{j=1}^{s} \frac{(-1)^{j-1} P_j(\log\log n)}{\log^{j+1} n} + o\left(\frac{1}{\log^{s+1} n}\right)\right]. \tag{4.7}$$

(2.1) gives

$$\tilde{\pi}(x) = \frac{x}{\log\sqrt{x} - 1 - \frac{k_1}{\log\sqrt{x}} - \frac{k_2}{\log^2\sqrt{x}} - \dots - \frac{k_{s+1}}{\log^{s+1}\sqrt{x}} + o\left(\frac{1}{\log^{s+1}\sqrt{x}}\right)}. \tag{4.8}$$

For $x = r_n$, this relation becomes (in view of (3.4)):

$$\frac{\sqrt{r_n}}{n} = \log\sqrt{r_n} - 1 - \frac{k_1}{\log\sqrt{r_n}} - \dots - \frac{k_{s+1}}{\log^{s+1}\sqrt{r_n}} + o\left(\frac{1}{\log^{s+1} n}\right). \tag{4.9}$$

If x and y are ≥ 1, Lagrange's theorem gives us the inequality

$$|\log y - \log x| \leq |y - x| ; \tag{4.10}$$

with (4.4) and (4.7), it leads to:

$$\log\sqrt{r_n} - \log p_n = o\left(\frac{1}{\log^{s+1} n}\right). \tag{4.11}$$

This last relation gives for every $t \in \{1, 2, \dots, s+1\}$

$$\frac{1}{\log^t p_n} - \frac{1}{\log^t \sqrt{r_n}} = o\left(\frac{1}{\log^{s+t+2} n}\right) = o\left(\frac{1}{\log^{s+1} n}\right). \tag{4.12}$$

(4.6), (4.9), (4.11) and (4.12) give

$$\frac{\sqrt{r_n} - p_n}{n} = o\left(\frac{1}{\log^{s+1} n}\right) \tag{4.13}$$

and the proof is complete. ☐

Theorem 4.2. *There exists a unique sequence $(R_m)_{m\geq 1}$ of polynomials with integer coefficients such that, for every positive integer m,*

$$r_n = n^2\left[\log^2 n + 2(\log\log n - 1)\log n + (\log\log n)^2 - 3 + \sum_{j=1}^{m} \frac{(-1)^{j-1} R_j(\log\log n)}{(j+1)! \cdot \log^j n}\right] + o\left(\frac{n^2}{\log^m n}\right). \tag{4.14}$$

Proof. (4.9) allows us to write

$$r_n = n^2\left[\log n + \log\log n - 1 + \sum_{j=1}^{m+1} \frac{(-1)^{j+1} Pj(\log\log n)}{j! \cdot \log^j n} + o\left(\frac{1}{\log^{m+1} n}\right)\right]^2. \tag{4.15}$$

If we set

$$R_1 := 4(X-1)P_1 - 2P_2 \tag{4.16}$$

and

$$R_j := -2P_{j+1} + 2(j+1)(X-1)P_j - \sum_{i=1}^{j-1}(j+1)\binom{j}{i}P_iP_{j-i} \quad ,j \geq 2 \tag{4.17}$$

(4.15) gives for every $m \geq 1$:

$$r_n = n^2\Bigg[\log^2 n + 2(\log\log n - 1)\log n + (\log\log n)^2 - 3 + \sum_{j=1}^{m}\frac{(-1)^{j-1}R_j(\log\log n)}{(j+1)!\cdot\log^j n}\Bigg] + o\left(\frac{n^2}{\log^m n}\right),$$

so the existence is proved.

Suppose now the existence of two different sequences $(R_m)_{m\geq 1}$ and $(S_m)_{m\geq 1}$ satisfying the conditions of the theorem. For the least j such as $S_j \neq R_j$ we can write

$$\frac{R_j(\log\log n) - S_j(\log\log n)}{(j+1)!\cdot\log^j n} = o\left(\frac{1}{\log^j n}\right),$$

so $R_j(\log\log n) - S_j(\log\log n) = o(1)$, a contradiction. □

Corollary 4.3. *We have*

$$r_n = n^2\log^2 n + 2n^2(\log\log n - 1)\log n + n^2(\log\log n)^2 - 3n^2 + o(n^2).$$

5. COMPUTING THE COEFFICIENTS OF THE POLYNOMIAL R_m

Proposition 5.1. *For every $m \geq 1$, the degree of R_m is $m+1$ and its leading coefficient is $2(m-1)!$.*

Proof. If we recall from the introduction that every P_n has degree n and leading coefficient $(n-1)!$, the statement follows from (4.16) and (4.17). □

(1.4) gives

$$P_1(X) = X - 2.$$

We can easily derive from M. Cipolla's paper [1] the relations

$$P_k' = k(k-1)P_{k-1} + k\cdot P_{k-1}' \quad ,k\geq 2$$

and

$$P_{k+1}(0) = -k\left\{\sum_{j=1}^{k-1}\binom{k-1}{j}P_j(0)[P_{k-j}(0) + P_{k-j}'(0)] + [P_k(0) + P_k'(0)]\right\} - (k+1)P_k(0) - P_{k+1}'(0).$$

Computations gave him

$P_2(X) = X^2 - 6X + 11;$
$P_3(X) = 2X^3 - 21X^2 + 84X - 131;$
$P_4(X) = 6X^4 - 92X^3 + 588X^2 - 1908X + 2666;$
$P_5(X) = 24X^5 - 490X^4 + 4380X^3 - 22020X^2 + 62860X - 81534;$
$P_6(X) = 120X^6 - 3084X^5 + 35790X^4 - 246480X^3 + 1075020X^2 - 2823180X + 3478014;$
$P_7(X) = 720X^7 - 22428X^6 + 322224X^5 - 2838570X^4 + 16775640X^3 - 66811920X^2$
$+ 165838848X - 196993194.$

In view of (4.16) and (4.17), we get in turn:

$R_1(X) = 2X^2 - 14;$
$R_2(X) = 2X^3 - 6X^2 - 42X + 172;$
$R_3(X) = 4X^4 - 24X^3 - 144X^2 + 1544X - 3756;$
$R_4(X) = 12X^5 - 110X^4 - 600X^3 + 12300X^2 - 64060X + 122298;$
$R_5(X) = 48X^6 - 600X^5 - 2940X^4 + 102000X^3 - 842520X^2 + 3319512X - 5484780;$
$R_6(X) = 240X^7 - 3836X^6 - 16380X^5 + 913080X^4 - 10543400X^3 + 63989100X^2 - 215203884X + 323035480.$

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