COMPOSITION ALGEBRAS, EXCEPTIONAL JORDAN ALGEBRA AND RELATED GROUPS*

IVAN TODOROV AND SVETLA DRENSKA

Abstract. Normed division rings are reviewed in the more general framework of composition algebras that include the split (indefinite metric) case. The Jordan-von Neumann-Wigner classification of finite dimensional Jordan algebras is outlined with special attention to the 27 dimensional exceptional Jordan algebra \(J\). The automorphism group \(F_4\) of \(J\) and its maximal Borel-de Siebenthal subgroups \(SU(3) \times SU(3) \times \mathbb{Z}_3\) and \(Spin(9)\) are studied in some detail with an eye to possible applications to the fundamental fermions in the Standard Model of particle physics.

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Contents

1 Introduction 60

2 Composition and Clifford Algebras 60
  2.1 Normed Alternative Algebras ........................................... 60
  2.2 Relation to Clifford Algebras and Classification ...................... 63
  2.3 Historical Note ............................................................ 66

3 Octonions. Isometries and Automorphisms 67
  3.1 Eight Dimensional Alternative Algebras ............................... 67
  3.2 Isometry Group of the (Split) Octonions. Triality ................... 68
  3.3 Automorphism Group and Derivations of Octonions .................. 69
  3.4 Roots and Weights of \(\mathfrak{g}_{2(2)} \subset \mathfrak{so}(4,4)\) ...................... 71

4 Jordan Algebras and Related Groups 73
  4.1 Classification of Finite Dimensional Jordan Algebras ............... 73
  4.2 Automorphism Groups of the Exceptional Jordan Algebras \(\mathcal{H}_3(\mathbb{O}_{(3)})\) and their Maximal Subgroups ............................ 76
  4.3 The Jordan Subalgebra \(JSpin_9\) of \(\mathcal{H}_3(\mathbb{O})\) and its Automorphism Group \(Spin(9) \subset F_4\) ................................. 80

* Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.