TWISTOR SPACES AND COMPACT MANIFOLDS ADMITTING BOTH KÄHLER AND NON-KÄHLER STRUCTURES

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Abstract. In this expository paper we review some twistor techniques and recall the problem of finding compact differentiable manifolds that can carry both Kähler and non-Kähler complex structures. Such examples were constructed independently by Atiyah, Blanchard and Calabi in the 1950’s. In the 1980’s Tsanov gave an example of a simply connected manifold that admits both Kähler and non-Kähler complex structures - the twistor space of a K3 surface. Here we show that the quaternion twistor space of a hyperkähler manifold has the same property.

MSC: 53C28, 32L25, 14J28, 53C26

Keywords: hyperkähler manifolds, K3 surfaces, twistor spaces

1. Introduction

In this paper we discuss a couple of classical approaches to twistor theory. Roughly speaking, the twistor space \( Z(M) \) is a family of (almost) complex structures on an orientable Riemannian manifold \((M,g)\) compatible with the given metric \(g\) and the orientation. We are going to apply twistor techniques towards the problem of constructing simply connected compact manifolds that carry both Kähler and non-Kähler complex structures.

Atiyah’s idea behind his examples in [1] was to consider the set of all complex structures \( M_n \) on the real torus \( T^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n} \) coming from the complex vector space structures on \( \mathbb{R}^{2n} \). The space \( M_n \) is a complex manifold which is differentially a product of an algebraic variety and the torus \( T^{2n} \), and therefore admits a Kähler structure. On the other hand, there exists a “twisted” complex structure on \( M_n \) which is non-Kähler. This rationale works in many other cases, and in particular, one can produce simply connected examples of similar nature. In [22] Tsanov showed that the twistor space of a K3 surface is a simply connected 6-dimensional compact manifold which carries both Kähler and non-Kähler complex structures. Here we give examples of twistor spaces of hyperkähler manifolds and show that they also carry both Kähler and non-Kähler complex structures.