THE 2-ADIC ORDER OF SOME GENERALIZED FIBONACCI NUMBERS

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Abstract

Let \( T_n = T_n(k) \) be the generalized Fibonacci sequence of order \( k \) defined by the recurrence \( T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}, n \geq k, \) with \( T_0 = 0 \) and \( T_1 = T_2 = \cdots = T_{k-1} = 1. \) In this paper, we fully and partially characterize the 2-adic valuations of \( T_n(4) \) and \( T_n(5), \) respectively. Moreover, we provide new addition formulas and congruences for the sequences \( \{ T_n(k) \}_{n \geq 0}. \)

1. Introduction

Let \( \{ F_n \}_{n \geq 0} \) be the Fibonacci sequence given by \( F_{n+2} = F_{n+1} + F_n, \) for \( n \geq 0, \) where \( F_0 = 0 \) and \( F_1 = 1. \) The \( p \)-adic order, \( \nu_p(r), \) of \( r \) is the exponent of the highest power of a prime \( p \) which divides \( r. \) The \( p \)-adic order of a Fibonacci number was completely characterized, see [4]. Much less is known about the generalized Fibonacci sequences. Let \( T_n = T_n(k), n \geq 0, \) denote the generalized Fibonacci sequence of order \( k \) defined by the recurrence relation

\[
T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}, n \geq k,
\]

and the initial conditions \( T_0 = 0, T_1 = T_2 = \cdots = T_{k-1} = 1. \) Note that sometimes the initial conditions are given by

\[
B_0 = B_1 = \cdots = B_{k-2} = 0, B_{k-1} = 1
\]

with \( B_n = B_n(k) \) while the recurrence \( B_n = B_{n-1} + B_{n-2} + \cdots + B_{n-k}, n \geq k, \) is preserved. By convention, we also set \( B_{-1}(k) = 0. \) Clearly, \( F_n = T_n(2) = B_n(2). \)

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Our goal is to present a systematic approach that helps establish the 2-adic order of $T_n(k)$, at least for some specialized sequences of the index $n$ (we point out that the 2-adic valuation of $T_n(3)$ was fully determined in [6]). Here, we focus on $T_n(k)$ for $k = 4$ and 5. The first few terms of the sequence $\{T_n(4)\}_{n \geq 0}$ are

$$0, 1, 1, 1, 3, 6, 11, 21, 41, 79, 152, 293, 565, 1089, 2099, 4046, 7799, \ldots$$

while those of $\{T_n(5)\}_{n \geq 0}$ are

$$0, 1, 1, 1, 4, 8, 15, 29, 57, 113, 222, 436, 857, 1685, 3313, 6513, \ldots$$

Our main results are Theorems 1, 2, and Lemmas 2, 5, and 6. We also suggest several conjectures, cf. Conjectures 1 and 2.

Throughout the paper, we emphasize the experimental aspects of finding and discovering relations, e.g., recurrence relations and congruences.

**Theorem 1.** For $n \geq 1$, we have

$$\nu_2(T_n(4)) = \begin{cases} 
0, & \text{if } n \not\equiv 0 \pmod{5}, \\
1, & \text{if } n \equiv 5 \pmod{10}, \\
\nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{10}.
\end{cases}$$

With $n \geq 1$ and $s \geq 1$ odd, this yields that

$$\nu_2(T_{5 \cdot 2^s}(4)) = n + 2.$$  \hfill (4)

We refer the reader to [2] for the 2-adic valuation of $B_n(4)$.

We make the following conjecture for the case of $k = 5$.

**Conjecture 1.** For $n \geq 1$, we have

$$\nu_2(T_n(5)) = \begin{cases} 
0, & \text{if } n \not\equiv 0 \text{ or } 5 \pmod{6}, \\
2, & \text{if } n \equiv 5 \pmod{12}, \\
1, & \text{if } n \equiv 11 \pmod{12}, \\
\nu_2(n + 2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n + 2) < 8, \\
\nu_2(n + 43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n + 2) \geq 8, \\
\nu_2(n), & \text{if } n \equiv 0 \pmod{12}.
\end{cases}$$

Here, we prove it in the following weaker form.

**Theorem 2.** For $n \geq 1$, we have

$$\nu_2(T_n(5)) = \begin{cases} 
0, & \text{if } n \not\equiv 0 \text{ or } 5 \pmod{6}, \\
2, & \text{if } n \equiv 5 \pmod{12}, \\
1, & \text{if } n \equiv 11 \pmod{12}, \\
\nu_2(n + 2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n - 6) \neq 3, \\
\nu_2(n), & \text{if } n \equiv 0 \pmod{12}.
\end{cases}$$

(6)
With \( n \geq 1 \) and \( s \geq 1 \) odd, this yields that
\[
\nu_2(T_{0 \cdot 2^n}(5)) = n + 1.
\]

(7)

We also propose the following conjecture.

**Conjecture 2.** For \( n \geq 1 \) and \( k \geq 2 \) integers and \( s \geq 1 \) odd integer, we have
\[
\nu_2(T_{s \cdot (k+1) \cdot 2^n}(k)) = n + c(k)
\]
where \( c(2) = 2 \) and otherwise,

\[
c(k) = \begin{cases} 2, & \text{if } k \equiv 0 \pmod{4}; \\ 1, & \text{if } k \equiv 1 \pmod{4}; \\ \nu_2(k - 2) + 1, & \text{if } k \equiv 2 \pmod{8}; \\ 1, & \text{if } k \equiv 3 \pmod{8}; \\ 3, & \text{if } k \equiv 6 \pmod{8}; \\ 1, & \text{if } k \equiv 7 \pmod{8}. \end{cases}
\]

**Remark 1.** Conjecture 2 can be easily verified for \( k = 2 \) and 3. In fact, we proved for \( n \geq 1 \) that \( \nu_2(T_n(2)) = \nu_2(n) + 2 \) if \( n \equiv 0, 6 \pmod{12} \) in [4] and \( \nu_2(T_n(3)) = \nu_2(n) - 1 \) if \( n \equiv 0, 8 \pmod{16} \) in [6]. In this paper, we prove the conjecture for \( k = 4 \) and 5.

We outline a plan that can be followed in order to prove Conjecture 2. In fact, we will apply the plan in the cases of \( k = 4 \) and 5 in Sections 4 and 5, respectively.

**Step 1.** First we establish an addition formula for \( T_{q+r}(k) \) in terms of \( T_{q'}(k) \) and \( T_{r'}(k) \) with \( q' \) and \( r' \) close to \( q \) and \( r \), respectively; more precisely, with \( q - k + 2 \leq q' \leq q + k \) and \( r \leq r' \leq r + k - 1 \).

**Step 2.** The second step is to come up with a set of induction hypotheses for \( T_{s \cdot (k+1) \cdot 2^n+i}(k) \pmod{2^{n+c(k)+1}} \) for all \( i : 0 \leq i \leq k - 1 \) and \( n \geq n_0(k) \) with some functions \( c(k) \) and \( n_0(k) \), e.g., \( T_{s \cdot (k+1) \cdot 2^n} \equiv s \cdot 2^{n+c(k)} \pmod{2^{n+c(k)+1}} \), \( n \geq 1 \), in Lemmas 5 and 6 and prove it simultaneously by using the recurrence relation for \( T_{q+r}(k) \) from the first step. Note that the congruence \( T_{s \cdot (k+1) \cdot 2^n+i}(k) \pmod{2^{n+c(k)+1}} \) will follow for any \( i \leq -1 \) and \( i \geq k \) by the recurrence (1).

**Step 3.** In the induction proof, first we deal with the case \( s = 1 \) and we prove this case by induction on \( n \). The same procedure will work for other values of \( s \).

In conclusion, this process yields that if \( m = s \cdot (k+1) \cdot 2^n \) and \( s \geq 1 \) is odd then \( \nu_2(T_m(k)) = n + c(k) \) for \( n \geq n_0(k) \).
We illustrate the actual steps in Sections 2 and 3. Section 2 is devoted to the process of obtaining recurrence relations while Section 3 contains the congruences that are the essential tools in proving Theorems 1 and 2.

The actual calculations and proofs in the cases of \( k = 4 \) and \( 5 \) are presented in Sections 4 and 5. They lead to identities (11) and (12) that are crucial in proving the congruences (14), (15), and (16).

2. Obtaining a Recurrence by an Addition Formula

As a reminder, we note the addition formula, given in Lemma 4 of [6], which yields a recurrence for \( T_{q+r}(3) \). For all integers \( q \) and \( r \) with \( q \geq 3 \) and \( r \geq 0 \), we have that

\[
T_{q+r} = T_{q-2}T_r + (T_{q-3} + T_{q-2})T_{r+1} + T_{q-1}T_{r+2}.
\]

Note that \( T_{q-1} = T_{q-4} + T_{q-3} + T_{q-2} \). It is determined in Theorem 2.1 of [7] that with \( T_n = T_n(4) \) and \( B_n = B_n(4) \), we have

\[
T_q = B_{q-2}T_1 + (B_{q-2} + B_{q-3})T_2 + (B_{q-2} + B_{q-3} + B_{q-4})T_3 + B_{q-1}T_4,
\]

for \( q \geq 5 \) where \( B_{q-1} = B_{q-2} + B_{q-3} + B_{q-4} + B_{q-5} \). The formula (8) can be easily generalized to

**Lemma 1.** For \( q \geq 5 \) and \( r \geq 0 \) with \( T_n = T_n(4) \) and \( B_n = B_n(4) \), we have that

\[
T_{q+r} = B_{q-2}T_{r+1} + (B_{q-2} + B_{q-3})T_{r+2} + (B_{q-2} + B_{q-3} + B_{q-4})T_{r+3} + B_{q-1}T_{r+4}.
\]

To obtain similar identities for a general \( k \), we use the fact that one can relate the sequences \( \{T_n(k)\}_{n \geq 0} \) and \( \{B_n(k)\}_{n \geq 0} \). In fact, we have the following general result

**Lemma 2.** Let \( k \geq 2 \) be an integer and set \( T_n = T_n(k) \) and \( B_n = B_n(k) \). For integers \( q > k \) and \( r \geq 0 \), we have that

\[
T_q = \sum_{i=1}^{k} \left( \sum_{j=2}^{i+1} B_{q-j} \right) T_i \quad \text{and} \quad T_{q+r} = \sum_{i=1}^{k} \left( \sum_{j=2}^{i+1} B_{q-j} \right) T_{r+i}.
\]

**Remark 2.** We also use identity (9) in its equivalent form

\[
T_{q+r} = \sum_{i=0}^{k-1} \left( \sum_{j=1}^{i+1} B_{q-j} \right) T_{r+i}.
\]

with \( q \geq k \geq 2 \) and \( r \geq 0 \), cf. (11) and (12).
We omit the proof which can be easily done by mathematical induction on \(q > k\) for every fixed \(r \geq 0\).

**Remark 3.** Identity (9) also works for sequences \(T_n(k)\) of real numbers satisfying (1) with arbitrary initial conditions.

Our next step is to determine \(B_q'(k)\) in (9) in terms of the sequence \(\{T_n(k)\}_{n \geq 0}\). We note that although \(B_{n+1}(3) = T_n(3)\), usually there is a non-trivial linear relationship between the two sequences. We use the approach outlined in [1]. The result is derived in (18) and (19) as well in (22) and (23), and used by (11) and (12) in Lemmas 3 and 4, respectively.

**Lemma 3.** For \(T_q+r(4)\) with \(q \geq 2\) and \(r \geq 0\), we have the recurrence

\[
T_{q+r} = \left(\frac{5}{3}T_q + \frac{1}{3}T_{q+1} + 2T_{q+2} - \frac{4}{3}T_{q+3}\right)T_r
+ \left(\frac{5}{3}T_{q-1} + 2T_q + \frac{7}{3}T_{q+1} + \frac{2}{3}T_{q+2} - \frac{4}{3}T_{q+3}\right)T_{r+1}
+ \left(\frac{5}{3}T_{q-2} + 2T_{q-1} + 4T_q + T_{q+1} + \frac{2}{3}T_{q+2} - \frac{4}{3}T_{q+3}\right)T_{r+2}
+ \left(\frac{5}{3}T_{q+1} + \frac{1}{3}T_{q+2} + 2T_{q+3} - \frac{4}{3}T_{q+4}\right)T_{r+3}.
\]

**Lemma 4.** For \(T_q+r(5)\) with \(q \geq 3\) and \(r \geq 0\), we have the recurrence

\[
T_{q+r} = \left(\frac{35T_q}{46} + \frac{11T_{q+1}}{23} + \frac{15T_{q+2}}{46} + \frac{18T_{q+3}}{23} - \frac{27T_{q+4}}{46}\right)T_r
+ \left(\frac{35T_{q-1}}{46} + \frac{57T_q}{46} + \frac{37T_{q+1}}{46} + \frac{51T_{q+2}}{46} + \frac{9T_{q+3}}{23} - \frac{27T_{q+4}}{46}\right)T_{r+1}
+ \left(\frac{35T_{q-2}}{46} + \frac{57T_{q-1}}{46} + \frac{36T_q}{23} + \frac{73T_{q+1}}{46} + \frac{12T_{q+2}}{23}
+ \frac{9T_{q+3}}{46} - \frac{27T_{q+4}}{46}\right)T_{r+2}
+ \left(\frac{35T_{q-3}}{46} + \frac{57T_{q-2}}{46} + \frac{36T_{q-1}}{23} + \frac{54T_q}{23} + T_{q+1}
+ \frac{12T_{q+2}}{23} + \frac{9T_{q+3}}{23} - \frac{27T_{q+4}}{46}\right)T_{r+3}
+ \left(\frac{35T_{q+1}}{46} + \frac{11T_{q+2}}{23} + \frac{15T_{q+3}}{46} + \frac{18T_{q+4}}{23} - \frac{27T_{q+5}}{46}\right)T_{r+4}.
\]
3. Congruences

We note that for \( k = 3 \) the congruences in (4) of Lemma 6 in [6] are equivalent to the following statement. For \( s \geq 1, n \geq 3, \) and \( T_m = T_m(3) \), we have the congruences

\[
\begin{align*}
T_{s \cdot 2^n} & \equiv s \cdot 2^{n-1} \pmod{2^n}, \\
T_{s \cdot 2^n+1} & \equiv 1 \pmod{2^n}, \\
T_{s \cdot 2^n+2} & \equiv 1 + s \cdot 2^{n-1} \pmod{2^n}.
\end{align*}
\]

(13)

Now we establish similar congruences for \( k = 4 \).

**Lemma 5.** For \( s \geq 1, n \geq 2, \) and \( T_m = T_m(4) \), we have that

\[
\begin{align*}
T_{5 \cdot s \cdot 2^n} & \equiv s \cdot 2^{n+2} \pmod{2^{n+3}}, \\
T_{5 \cdot s \cdot 2^n+1} & \equiv 1 + s \cdot 2^{n+1} \pmod{2^{n+3}}, \\
T_{5 \cdot s \cdot 2^n+2} & \equiv 1 + s \cdot 2^{n+1} + s \cdot 2^{n+2} \pmod{2^{n+3}}, \\
T_{5 \cdot s \cdot 2^n+3} & \equiv 1 \pmod{2^{n+3}},
\end{align*}
\]

(14)

while for \( n = 1 \), we have that

\[
\begin{align*}
T_{10 \cdot s} & \equiv 8s \pmod{16}, \\
T_{10 \cdot s+1} & \equiv 1 + 4s \pmod{16}, \\
T_{10 \cdot s+2} & \equiv 1 + 4s \pmod{16}, \\
T_{10 \cdot s+3} & \equiv 1 \pmod{16},
\end{align*}
\]

(15)

which yields that \( \nu_2(T_{5 \cdot s \cdot 2^n}(4)) = n + 2 \) if \( n \geq 1 \) and \( s \geq 1 \) odd.

**Proof of Lemma 5.** We closely follow the steps of the proof of Lemma 6 of [6]. First, we deal with the basis case \( s = 1 \). We have to prove (14) for \( n \geq 2 \). We use induction on \( n \). Clearly, the congruences hold for \( n = 2 \). We suppose that they are true for \( n \geq 2 \), and then we use (11) for \( T_{5 \cdot 2^n+i+1} = T_{(5 \cdot s \cdot 2^n)+(5 \cdot 2^n+i)}, 0 \leq i \leq 3 \), to obtain the required congruences for \( T_{5 \cdot 2^n+i+1} \). Next, by the induction hypothesis, we suppose that the congruences (14) hold for \( s \geq 1 \). Then, we use exactly the same procedure and (11) as before for \( T_{5 \cdot (s+1) \cdot 2^n+i} = T_{(5 \cdot s \cdot 2^n)+(5 \cdot 2^n+i)} \). In a similar fashion, we use induction on \( s \geq 1 \) to prove the congruences (15), corresponding to the case with \( n = 1 \). We omit the details. \( \square \)

**Example 1.** We illustrate the above proof in the case of \( k = 4, n \geq 2, s \geq 1, \) and \( i = 0 \). With the setting \( r = 5 \cdot s \cdot 2^n \) and \( q = 5 \cdot 2^n \), we obtain by (11) that

\[
T_{5 \cdot 2^n(s+1)} = \left( \frac{5}{3} T_{5 \cdot 2^n} + \frac{1}{3} T_{5 \cdot 2^n+1} + 2T_{5 \cdot 2^n+2} - \frac{4}{3} T_{5 \cdot 2^n+3} \right) T_{5 \cdot 2^n} + \left( 2T_{5 \cdot 2^n} + \frac{2}{3} T_{5 \cdot 2^n-1} + \frac{7}{3} T_{5 \cdot 2^n+1} + \frac{2}{3} T_{5 \cdot 2^n+2} \right) T_{5 \cdot 2^n} 
\]

...
Using the results of Lemma 1, we obtain
\[
\frac{4}{3}T_{5,2^{n+1}} + \left( \frac{2}{3}T_{5,2^n+1} + \frac{1}{3}T_{5,2^n+2} + 2T_{5,2^n+3} - \frac{4}{3}T_{5,2^n+4} \right)T_{5,2^n+3},
\]
which results in
\[
\frac{1}{3}2^{2n+2}s - \frac{1}{3}2^{n+3}s + \frac{1}{3}2^{n+4}s + \frac{2}{3}2^{2n+3}s + 2^{2n+4}s + \frac{1}{3}2^{2n+6}s + \frac{1}{3}2^{2n+7}s - \frac{1}{3}2^{2n+8}s + \frac{1}{3}2^{2n+9}s + \frac{2}{3}2^{n+3} \quad \pmod{2^{n+3}}
\]
by the induction hypothesis. We get
\[
\frac{1}{3} \cdot 2^{n+2} \cdot (s + 1) \equiv 2^{n+2} \cdot (s + 1) \quad \pmod{2^{n+3}}
\]
by replacing any term including a factor with a “high” power of 2 with 0. More precisely, any term including \(2^{c-n+d}\) with \(d \geq 3\) or \(c > 1\) combined with \(d \geq 1\) is dropped. It implies that the statement
\[
T_{5,(s+1)2^n} \equiv (s+1) \cdot 2^{n+2} \quad \pmod{2^{n+3}}
\]
in (14) is also true.

Note that the substitutions and simplifications above can be easily preformed by using Mathematica.

In the case of \(k = 5\) we proceed similarly.

**Lemma 6.** For \(s \geq 1\), \(n \geq 1\), and \(T_m = T_m(5)\), we have that
\[
\begin{align*}
T_{6,s \cdot 2^n} &\equiv s \cdot 2^{n+1} \quad \pmod{2^{n+2}}, \\
T_{6,s \cdot 2^n+1} &\equiv 1 \quad \pmod{2^{n+2}}, \\
T_{6,s \cdot 2^n+2} &\equiv 1 + s \cdot 2^{n+1} \quad \pmod{2^{n+2}}, \\
T_{6,s \cdot 2^n+3} &\equiv 1 \quad \pmod{2^{n+2}}, \\
T_{6,s \cdot 2^n+4} &\equiv 1 \quad \pmod{2^{n+2}},
\end{align*}
\]
which yields that \(\nu_2(T_{6,s \cdot 2^n}(5)) = n + 1\) if \(n \geq 1\) and \(s \geq 1\) odd.

The proof essentially duplicates the steps of the proof of Lemma 5 and we leave the details to the reader.

### 4. The Case of \(k = 4\)

Before we present the proof of Lemma 3, we explore an approach given in [1]. In fact, we use it with some modifications and with \(n \geq 0\) and \(m \geq 4\). We start with the matrix
\[
\begin{pmatrix}
T_n & T_{n+1} & T_{n+2} & T_{n+3} & T_{m+n} \\
T_{n+1} & T_n & T_{n+2} & T_{n+3} & T_{m+n+1} \\
T_{n+2} & T_{n+1} & T_n & T_{n+2} & T_{m+n+2} \\
T_{n+3} & T_{n+2} & T_{n+1} & T_n & T_{m+n+3} \\
T_{m+n} & T_{m+n+1} & T_{m+n+2} & T_{m+n+3} & T_{m+n+4}
\end{pmatrix}
\]

(17)

After experimenting with different values of \(m\) and row reducing the matrix in (17), we successfully obtain the recurrence relation \(T_{m+n} = B_{m-1}T_n + (B_{m-2} + B_{m-1})T_{n+1} + (B_{m-3} + B_{m-2} + B_{m-1})T_{n+2} + B_{m-3}T_{n+3}\) suggesting (9) of Lemma 2 in its equivalent form (10) for \(k = 4\) with \(m \geq 4\) and \(n \geq 0\).

In a similar fashion, we establish the
Proof of Lemma 3. We consider the matrix

\[
\begin{pmatrix}
T_n & T_{n+1} & T_{n+2} & T_{n+3} & B_{m+n} \\
T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & B_{m+n+1} \\
T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & B_{m+n+2} \\
T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & B_{m+n+3}
\end{pmatrix}.
\]

(18)

After setting \( m = -1 \) and using different values of \( n \geq 1 \), we observe that the row reduction always results in

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 5 \\
0 & 1 & 0 & 0 & \frac{7}{3} \\
0 & 0 & 1 & 0 & \frac{2}{3} \\
0 & 0 & 0 & 1 & -\frac{4}{3}
\end{pmatrix},
\]

(19)

which yields that

\[
B_{n-1} = \left( \frac{5T_n}{3} + \frac{T_{n+1}}{3} + 2T_{n+2} - \frac{4T_{n+3}}{3} \right)
\]

(20)

for \( n \geq 1 \), which confirms (11).

Note that once (20) is established, an easy induction proof justifies this identity. Indeed, with \( n = 1, 2, 3, 4 \) we get that \( 0 = \frac{5}{3} \cdot 1 + \frac{1}{3} \cdot 1 + 2 \cdot 1 - \frac{4}{3} \cdot 3 = \frac{5}{3} \cdot 1 + \frac{1}{3} \cdot 1 + 2 \cdot 3 - \frac{4}{3} \cdot 6 = \frac{5}{3} \cdot 1 + \frac{1}{3} \cdot 3 + 2 \cdot 6 - \frac{4}{3} \cdot 11 \) and \( 1 = \frac{5}{3} \cdot 3 + \frac{1}{3} \cdot 6 + 2 \cdot 11 - \frac{4}{3} \cdot 21 \). The induction step is trivial by (1) and (2).

A natural approach to obtain the proof of Theorem 1 is to utilize the periodicity of the underlying sequences. In some cases we can apply multisection techniques, cf. [5], to find the complete or some partial characterization of the \( p \)-adic order of the sequences. Here we combine these methods with the applications of sets of congruences for \( \{T_s(k+1)2^s+1\}_{i=0}^{k-1} \) with \( s \geq 1 \) and \( n \geq n_0(k) \) integers.

Now we can complete the proof of Theorem 1.

Proof of Theorem 1. The proof for the case \( n \not\equiv 0 \pmod{5} \) is trivial by taking \( T_n(4) \pmod{2} \) and induction on \( n \). In fact, the sequence \( \{T_n(4)\}_{n \geq 0} \) is periodic with period \( \{0, 1, 1, 1\} \) modulo 2.

If \( n \equiv 5 \pmod{10} \) then by 5-section of the generating function \( \sum_{m=0}^{\infty} T_m(4)x^m \) (cf. [5]) we get that

\[
\sum_{m=0}^{\infty} T_{5m}(4)x^{5m} = \frac{2x^5(3-2x^5-x^{10})}{1-26x^5-16x^{10}-6x^{15}-x^{20}},
\]

which easily yields that \( \nu_2(T_n(4)) = 1 \). Indeed, the denominator of the 5-sected generating function suggests the recurrence

\[
T_{5m+10} = 26T_{5m+5} + 16T_{5m} + 6T_{5m-5} + T_{5m-10}, m \geq 2,
\]

(21)
for $T_r = T_r(4)$ with $r$ divisible by 5. We observe that $\nu_2(T_5) = 1$, $\nu_2(T_{10}) = 3$, $\nu_2(T_{15}) = 1$, and $\nu_2(T_{20}) = 4$, which yield that $\nu_2(T_{5m}) \geq 1$ for $m \geq 0$ by the initial values and (21). Now $\nu_2(T_{5m+10}) = \nu_2(T_{5m-10}) = 1$ with $m \geq 3$ odd also follows by recurrence (21).

We note that we can extend (15) by recurrence (1) to obtain $T_{10s+4} \equiv 3 \pmod{16}$ and $T_{10s+5} \equiv 6 + 8s \pmod{16}$, and the latter congruence also results in $\nu_2(T_n) = 1$ with $n \equiv 5 \pmod{10}$.

In the remaining case 10 divides $n$, and Lemma 5 concludes the proof. \hfill \Box

5. The Case of $k = 5$

Now we turn to the

Proof of Lemma 4. Similarly to (18) in the case of $k = 4$, we now consider

$$
\begin{pmatrix}
T_n & T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & B_{m+n} \\
T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & B_{m+n+1} \\
T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & B_{m+n+2} \\
T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & T_{n+7} & B_{m+n+3} \\
T_{n+4} & T_{n+5} & T_{n+6} & T_{n+7} & T_{n+8} & B_{m+n+4}
\end{pmatrix}
$$

(22)

After setting $m = -1$ and using different values of $n \geq 1$, row reduction leads us to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 35 \frac{1}{2} \\
0 & 1 & 0 & 0 & 0 & \frac{1}{1} \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1 & 0 & \frac{1}{5} \\
0 & 0 & 0 & 0 & 1 & \frac{1}{28}
\end{pmatrix}
$$

(23)

which results in $B_{n-1} = \frac{35T_n}{46} + \frac{11T_{n+1}}{23} + \frac{15T_{n+2}}{46} + \frac{18T_{n+3}}{23} - \frac{27T_{n+4}}{46}$ for $n \geq 1$, which is in agreement with (12). Its proof follows easily by induction as it was explained in the proof of Lemma 3 for $k = 4$. \hfill \Box

We are now ready to present the proof of Theorem 2.

Proof of Theorem 2. As above, the proof for the case $n \not\equiv 0 \pmod{6}$ is trivial by taking $T_n(5)$ (mod 2) and induction on $n$ since the sequence $\{T_n(5)\}_{n \geq 0}$ is periodic with period $\{0, 1, 1, 1, 1, 0\}$ modulo 2.

If $n \equiv 6 \pmod{12}$ then with $n = 6 \cdot s \cdot 2^m + 6$, $s \geq 1$ odd and $m \geq 1$, we get that $T_{6s, 2^m+5} \equiv 4 \pmod{2^{m+2}}$ and $T_{6s, 2^m+6} \equiv 8 + s \cdot 2^{m+1} \pmod{2^{m+2}}$ by extending (16) via (1). It implies that $\nu_2(T_{6s, 2^m+6}) = \nu_2(n + 2)$ as long as either $m \geq 3$ or $m = 1$, in which cases the 2-adic order is either 3 or 2, respectively. In a similar fashion, it follows that $T_{6s, 2^{m+11}} \equiv 222 \pmod{2^{m+2}}$. Thus, with $t \geq 1$ integer, we
also have that $T_{12t+5} \equiv 4 \pmod{8}$ and $T_{12t+11} \equiv 222 \pmod{8}$, which yield that $\nu_2(T_{12t+5}) = 2$ and $\nu_2(T_{12t+11}) = 1$.

Otherwise 12 divides $n$, and Lemma 6 concludes the proof. \hfill \Box

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References


