



AN UPDATE ON DOMINEERING ON RECTANGULAR BOARDS**Gabriel C. Drummond-Cole¹**

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Abstract

Domineering is a combinatorial game played on a subset of a rectangular grid between two players. Each board position can be put into one of four outcome classes based on who the winner will be if both players play optimally. In this note, we review previous work, establish the outcome classes for several dimensions of rectangular board, and restrict the outcome class in several more. Several types of computations are made to establish outcome classes for boards of certain fixed small widths; in addition there are new results for boards of dimension $n \times 2n$ for arbitrary n as well as a new approach with modest results to plumb the behavior of larger boards.

1. Introduction

Domineering, invented by Göran Andersson and introduced to the public by Martin Gardner [7], is a game played on a rectangular grid of squares between two players. The players take turns placing dominoes on unoccupied squares of the board. Each domino covers two adjacent squares. One player, called Vertical, must place her dominoes in a “vertical” orientation. The other, called Horizontal, places hers in a “horizontal” orientation. When a player has no legal move on her turn, she loses. We shall refer to the board with vertical dimension m and horizontal dimension n as the $m \times n$ board or as $G_{m,n}$.

Because this is a finite drawless turn-based perfect information game of no chance, each board position has a particular outcome class which describes the winner if neither player makes a mistake. This outcome class could be V if the vertical player will win, H if the horizontal player will win, N if the next player to move will win, regardless of whether it is vertical or horizontal, and P if the next player to move

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will lose.

Combinatorial game theory goes further, giving each board position a value in a partially ordered Abelian group \mathcal{G} [2]. The disjoint union of board positions corresponds to addition in the group. Outcome classes can be read off from values in the group: the outcome class of a board position is V if and only if the board position is greater than 0, is H if and only if the board position is less than 0, is N if and only if the board position is incomparable to 0, and is P if and only if the board position is equal to 0. Much of the domineering literature uses the notation 1 and 2 instead of N and P ; most combinatorial game theory literature uses the notation L and R instead of V and H .

Domineering has been studied both by mathematicians working in combinatorial game theory and by computer scientists working in artificial intelligence. Typically the computer scientists have been concerned exclusively or primarily with outcome classes while the mathematicians have been interested not only in outcome classes but also in other questions about \mathcal{G} values.

Berlekamp engaged in the first systematic research into the outcome classes of rectangular boards [1], giving precise \mathcal{G} values for the boards $G_{2,2k+1}$.

Breuker, Uiterwijk, and van den Herik [3] and Uiterwijk and van den Herik [11] used a computer program called DOMI that employed α - β pruning to determine the outcome classes of several boards. Their most recent publicly available results are in [9].²

Lachmann, Moore, and Rapaport [8] extended this work by means of several simple rules which allowed them to combine outcome classes of smaller boards to give outcome classes for some boards with one small dimension. They also gave a result for boards where the height is an integral multiple of the width or vice versa.

Bullock [4] wrote a computer program called Obsequi that employed α - β pruning to determine outcome classes of Domineering positions. This program had a number of performance enhancements over DOMI and was able to determine outcome classes for larger boards.

This paper uses a mixture of methods. We use Obsequi to analyze the disjoint union of rectangular positions with small non-rectangular positions of known \mathcal{G} -value to establish bounds in \mathcal{G} for rectangular positions. We also employ precise \mathcal{G} values for boards of the form $G_{n,2}$ for certain n , calculated with Berlekamp's formula or Siegel's cgsuite software [10]. We investigate the implications of the methods of Lachmann et al. and apply these methods to the calculations made with Obsequi and cgsuite. We also improve one of their methods for boards where the height is an even integral multiple of the width (or vice versa). All of this allows the following

²They employ the convention that V moves first so their outcome class 1 is the same as our outcome class NV and their outcome class 2 is the same as our outcome class PH . Using symmetry about the diagonal, outcome classes in $\{N, P, H, V\}$ for the sizes they analyze can be recovered from their table.

previously unpublished results:

Results.

1. *The outcome class of $G_{6,n}$ is N for $n \in \{15, 19\}$;*
2. *The outcome class of $G_{6,n}$ is H for $n \in \{41, 49, 53, 55, 57\}$ and all odd $n > 59$;*
3. *The outcome class of $G_{6,n}$ is either N or H for $n \equiv 1 \pmod{4}$, $n > 21$;*
4. *The outcome class of $G_{8,n}$ is H for $n \in \{26, 30, 36, 40, 42, 46, 48, 50, 52\}$ and all even $n > 54$;*
5. *The outcome class of $G_{8,n}$ is either N or H for $n \in \{28, 34, 38, 44, 54\}$;*
6. *The outcome class of $G_{9,n}$ is H for $n \in \{13, 15, 17, 19, 21\}$;*
7. *The outcome class of $G_{11,n}$ is H for $n \in \{14, 18\}$ and for odd n greater than 31;*
8. *the outcome class of $G_{15,n}$ is either N or H for $n \in \{10, 14, 18\}$;*
9. *The outcome class of $G_{n,2kn}$ is H for all n and k .*

The same results hold with V substituted for H and the two indices of G in the reverse order.

From a computational perspective, finding results on individual boards is an intrinsically interesting challenge in optimization. From a mathematical perspective, it may be more interesting to find results on families of boards. The tools of Lachmann et al. provide methods which in some cases can extend results about individual boards to results about other boards of the same height or width. But these tools can give no information for boards where both dimensions are large. In particular, our results for $G_{15,n}$ are essentially the first results for a board where both the height and width are greater than thirteen.

In the service of generating knowledge about large boards, one can interpret results for small boards as being not just of computational interest but also as evidence of patterns in the outcome classes that may hold in the large limit. Previous researchers have taken this point of view, making conjectures about large boards. The final section of the paper provides weak results concerning these conjectures. The final result in the list of results above applies to large boards and can be exploited to prove some non-existence results that do not give explicit results for any fixed board size.

2. Results on Individual Boards

Lachmann et al. use a number of tools to combine results for smaller boards into results for larger boards. The simplified versions of their theorems that we will use are the following. We will use the notation $|G_{m,n}|$ to denote the \mathcal{G} value of $G_{m,n}$.

Proposition 1 (The one-hand-tied principle for rectangular boards).

$$|G_{m,n_1+n_2}| \leq |G_{m,n_1}| + |G_{m,n_2}|.$$

The applications are as follows: if the outcome class of G_{m,n_1} is H and the the outcome class of G_{m,n_2} is N (respectively P or H) then the outcome class of G_{m,n_1+n_2} is either N or H (respectively H).

Proposition 2. *The outcome class of $G_{m,2km}$ is P or H .*

We shall improve this result with Proposition 11, but the version of Lachmann et al. suffices for the results of this section.

We can directly combine these with Bullock’s result that the outcome class of $G_{8,10}$ is H to obtain the following proposition, which can be viewed as an application of the general principle of Proposition 13.

Proposition 3. *The outcome class of $G_{8,n}$ is H for $n \in \{26, 30, 32, 36, 40, 42, 46, 48, 50, 52\}$ and all even $n > 54$, and is either N or H for $n \in \{28, 34, 38, 44, 54\}$.*

Proof. Using the one-hand-tied principle (Proposition 1), since the outcome class of $G_{8,10}$ is H , it suffices to prove the proposition for $G_{8,n}$ where n ranges over the smallest representative of each residue class modulo ten in the claimed set of n . That is, we need only show the theorem for $n \in \{26, 28, 32, 34, 48, 64\}$. In each case, we will use the one-hand-tied principle, combining $G_{8,10}$ with $G_{8,8}$ (outcome class 1) and/or $G_{8,16}$ (outcome class H by Proposition 11).

Since $26 = 10 + 16$, $G_{8,26}$ has outcome class H . Since $28 = 2 \times 10 + 8$, $G_{8,28}$ has outcome class N or H . Similarly, $34 = 10 + 16 + 8$. The widths 32, 48, and 64 are integer multiples of 16 which suffices to show that the boards of that width have outcome class H . Without using the new result of Proposition 11, similar but slightly more intricate arguments could still show the result for $G_{8,48}$ and $G_{8,64}$. \square

A number of boards can be analyzed by looking at the exact \mathcal{G} values of $G_{n,2}$. The notation and definition of addition in \mathcal{G} can be found in [2].

Proposition 4.

1. *The outcome classes of $G_{11,14}$ and $G_{11,18}$ are H ;*
2. *The outcome class of $G_{15,n}$ is either N or H for $n \in \{6, 10, 14, 18\}$;*

3. The outcome class of $G_{19,6}$ is N .

Proof. Using Berlekamp’s formula or Siegel’s cgsuite software, we can determine that

$$|G_{11,2}| = \{1||\frac{1}{2}| - 1|| - \frac{3}{2}| - \frac{7}{2}\}.$$

Using the one-hand-tied principle, we know that

$$|G_{11,14}| \leq 7|G_{11,2}| = \{2|0|| - \frac{1}{2}| - 2|| - \frac{5}{2}\} < 0$$

so $G_{11,14}$ has outcome class H . The one-hand-tied principle gives us the same for $G_{11,18}$.

Similarly,

$$|G_{15,2}| = \{3|\frac{3}{2}||1| - \frac{1}{2}|| - 1\}$$

and it is not hard to check that

$$\begin{aligned} 3|G_{15,2}| &= \{\frac{3}{2}||1| - \frac{1}{2}|| - 1| - \frac{5}{2}\}, \\ 5|G_{15,2}| &= \{\frac{7}{2}|2|\frac{3}{2}|0|| - \frac{1}{2}\}, \\ 7|G_{15,2}| &= \{2||\frac{3}{2}|0|| - \frac{1}{2}| - 2\}, \text{ and} \\ 9|G_{15,2}| &= \{4|\frac{5}{2}||2|\frac{1}{2}||0\}. \end{aligned}$$

are all incomparable with zero, so that the outcome classes of the corresponding boards are either N or H .

Finally,

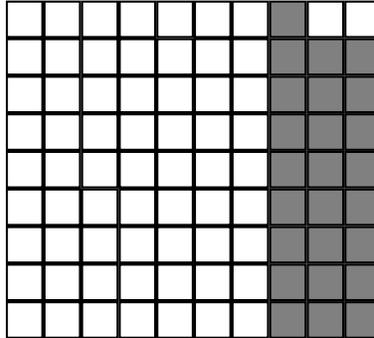
$$|G_{19,2}| = \{\frac{3}{2}||1| - \frac{1}{2}|| - 1| - \frac{5}{2}\}$$

and

$$3|G_{19,2}| = \{4|\frac{5}{2}||2|\frac{1}{2}||0\}$$

so the outcome class of $G_{19,6}$ is either N or H . Lachmann et al. determined that the outcome class of that board was either N or V , so it must be N . \square

Using computer tools such as Obsequi, one can establish bounds on the \mathcal{G} values of Domineering rectangles in terms of games that can be represented by simple Domineering positions. For example, to verify that $|G_{9,7}| \leq 1$, we can test whether $|G_{9,7}| + (-1) \leq 0$, that is, whether the following game has outcome class either P or H (here gray squares are unplayable):



This is precisely the sort of problem that Obsequi is equipped to handle, and it verifies that indeed, $|G_{9,7}| \leq 1$.

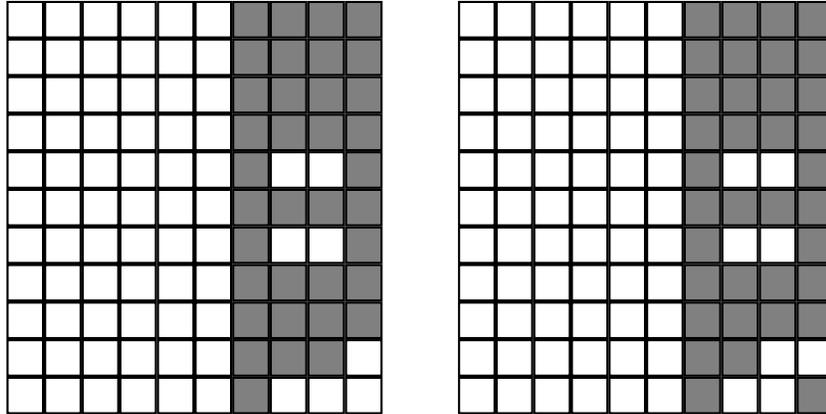
Since this bound and the others for the following proposition are rough, it is possible that similar methods could establish the outcome class of $G_{9,11}$ (the only outstanding board of height 9) and/or boards of height six and eleven that are too large for cgsuite to feasibly analyze given current computational resources.

Proposition 5.

1. *The outcome class of $G_{9,n}$ is H for $n \in \{13, 15, 17, 19, 21\}$;*
2. *The outcome class of $G_{11,n}$ is H for odd n greater than 31;*
3. *The outcome class of $G_{6,15}$ is N ;*
4. *The outcome class of $G_{6,41}$ is H .*

Proof. Since $G_{9,2}$ has outcome class H , for the first part it suffices to check $G_{9,13}$. As in Proposition 4, we can verify by cgsuite that $3|G_{9,2}| = \{1|-1||-\frac{3}{2}|-3\}$. Above, we described using Obsequi to verify that $|G_{9,7}| \leq 1$. Then by the one-hand-tied principle, $|G_{9,13}| \leq 3|G_{9,2}| + |G_{9,7}| \leq \{2|0||-\frac{1}{2}|-2\} < 0$, so $G_{9,13}$ has outcome class H .

For the second part, it suffices to check for $G_{11,33}$ and $G_{11,35}$. We make two distinct verifications: that $|G_{11,5}| \leq \frac{5}{2}$ and that $|G_{11,5}| \leq \{3|2\}$. Obsequi can demonstrate these by verifying that H wins if V goes first on the following two positions:



Using the one-hand-tied principle we get

$$|G_{11,33}| \leq 14|G_{11,2}| + |G_{11,5}| \leq \left\{ -\frac{1}{2} \left| -\frac{5}{2} \right| - 3 \left| -\frac{9}{2} \right| \right\} + \frac{5}{2} < 0$$

and

$$|G_{11,35}| \leq 15|G_{11,2}| + |G_{11,5}| \leq \left\{ \frac{1}{2} \left| -\frac{3}{2} \right| - 2 \left| -\frac{7}{2} \right| \right\} - 4 + \{3|2\} < 0$$

which completes the proof.

For the third part, we can use Obsequi as above to demonstrate a slightly different kind of inequality. We can add the infinitesimal \uparrow to $G_{6,11}$. Obsequi verifies that this sum has outcome class N . $G_{6,4}$ has a complicated \mathcal{G} -value but it can be verified (using the value of $G_{6,2}$, say) that $|G_{6,4}| < \uparrow$. Therefore by the one-hand-tied argument, the outcome class of $G_{6,15}$ is either N or H . We have already shown in Proposition 4 that it is either N or V , so it is N^3 .

For the final part, we can use Obsequi to verify that $|G_{6,12}| \leq -\frac{1}{2}$. We can combine this with the \mathcal{G} -value calculation that $|G_{6,5}| = \frac{3}{2}$ and the one-hand-tied principle for $41 = 3 \times 12 + 5$ to show that the outcome class of $G_{6,41}$ must be either P or H . Using the one-hand-tied principle with $41 = 2 \times 8 + 14 + 11$, we see that the outcome class must be either N or H . Therefore it is H . \square

Corollary 6.

1. The outcome class of $G_{6,n}$ is either N or H for $n > 21$ and
2. The outcome class of $G_{6,n}$ is H for $n \in \{49, 53, 55, 57\}$ and all $n > 59$,

³Bullock recently verified [6] that $G_{6,15}$ is either N or H by using Obsequi to evaluate the rectangle of size 6×15 directly, employing an updated version of Obsequi as well as more patience and computing power than the author.

Proof. For the first part, using the one-hand-tied principle, since the outcome class of $G_{6,8}$ is H , it suffices to prove the corollary for $G_{6,n}$ where n ranges over representatives of each residue class modulo eight which are no greater than 29. Lachmann et al. showed this for all residues except 1 and 5. Bullock showed it for the residue 1. So it suffices to show that $G_{6,29}$ has outcome class N or H . The one-hand-tied principle, using $29 = 14 + 15$ and the new result that the outcome class of $G_{6,15}$ is N , completes the argument.

The second part is a direct corollary of the fact that the outcome class of $G_{6,41}$ is H , using the one-hand-tied principle and what is already known about outcome classes of the boards $G_{6,n}$. \square

3. Results for Larger Boards

It is evident by symmetry that square boards must have outcome class either N or P . Exploiting this symmetry, Lachmann et al.'s Proposition 2 states that boards of the form $G_{m,2km}$ have outcome class either P or H and further implies that boards of the form $G_{m,2km+m}$ do not have outcome class V . Using Proposition 1 and known results about boards of width two, Lachmann et al. could also show that boards of the form $G_{m,2km+2}$ have outcome class N or H for $m \in \{14, 15, 18, 19, 23, 27\}$. Other than these results, nothing was known about outcome classes for boards with both dimensions greater than 13. Proposition 4 slightly extends the horizon of the unknown by establishing that $G_{15,14}$, $G_{15,18}$, $G_{15+30k+i}$ ($i \in \{6, 10, 14, 18\}$), and $G_{19+38k+6}$ have outcome class N or H .

Although there are virtually no methods or results for large boards, there has been conjecture as to what the outcome classes of large boards may be. The first two conjectures are from [11] translated into the language of this paper; the third and fourth are from [8]

Conjecture 7 (Odd height wide board conjecture). *For m odd, $G_{m,n}$ has outcome class H for all $n > m$.*

Conjecture 8 (Width divisible by four wide board conjecture). *For all m , $G_{m,n}$ has outcome class H for all n divisible by four such that $n > m$.*

Conjecture 9 (Bounded away from zero conjecture). *For all m there exists n such that the \mathcal{G} value of $G_{m,n}$ is less than some negative number.*

Lachmann et al. note that this implies the following:

Conjecture 10 (Weak wide board conjecture). *For all m there exists an M_m such that $G_{m,n}$ has outcome class H for all $n > M_m$.*

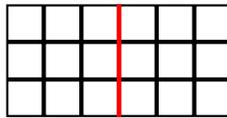
There are many potential variations of these; further conjecture is left to the reader's imagination.

The results of this section are perhaps best viewed as modest evidence in favor of the various wide board conjectures. There is one concrete result for large boards, the following improvement of Proposition 2.

Proposition 11. *For all positive m and k , the outcome class of $G_{m,2km}$ is H .*

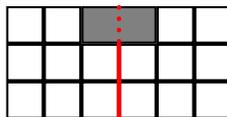
Proof. It suffices to show the result for $G_{m,2m}$.

Lachmann et al. show that this outcome class must be P or H using the one-hand-tied principle. Their proof is as follows. A proof of the one-hand-tied principle is that the horizontal player can only hurt her outcome class by refusing to move across the red line:



Because the position where the horizontal player does not move across the red line is two copies of a position which is invariant under ninety degree rotations, if the horizontal player goes second, she can copy the vertical player's moves on one half, rotating them on the other. So the position where the horizontal player refuses to move across the red line has outcome class P , meaning the original position has outcome class P or H .

On the other hand, if the horizontal player goes first on the original board, she can move across the red line once at the beginning and then refuse to do so from then on, leaving the following position:



The right half of this position is a ninety degree rotation of the left half. So after the horizontal player has made her first move, she can refuse to move across the red line and copy the vertical player's moves as in the proof of Lachmann et al. This shows that the horizontal player can win going first. \square

Corollary 12. *For all positive m and k , the outcome class of $G_{m,(2k+1)m}$ is N or H .*

This proposition implies that for fixed board height, \mathcal{G} values of the game are bounded above and decrease when the board width grows. The structure of the proof implies that $G_{m,2m} < 0$ but it does not give a much better upper bound on the \mathcal{G} value of $G_{m,2m}$, only some negative infinitesimal relatively close to zero, so it does not yield the bounded away from zero conjecture.

The final results of this paper do not yield explicit outcome classes for $G_{m,n}$ for any explicit m or n but rather put into place existence restrictions (i.e. Propositions 15 and 17).

Proposition 13. *Suppose that the outcome class of $G_{m,j}$ is H and that the outcome class of $G_{m,k}$ is P or H . Then for sufficiently high N , the outcome class of $G_{m,M \gcd(j,k)}$ is H .*

Proof. Schur's theorem says that only finitely many positive multiples of $\gcd(j, k)$ cannot be expressed as a sum of the form $aj + bk$ with nonnegative a and b . If $M \gcd(j, k) > jk$, then any such expression can be modified so that a is strictly positive. Then the one-hand-tied principle implies the result. \square

Corollary 14. *Suppose that the outcome class of $G_{m,j}$ is P or H . Then for sufficiently high M , the outcome class of $G_{m,M \gcd(j,2m)}$ is H .*

Proposition 15. *For a fixed height m , only finitely many of the boards $G_{m,n}$ have outcome class P .*

Proof. If $G_{m,k}$ has outcome class P then $G_{m,2mi+k}$ has outcome class H . So there can be at most $2m$ boards of height G and outcome class P . \square

Lemma 16. *Suppose that for a fixed height m , infinitely many of the boards $G_{m,n}$ have outcome class V . Then there exists a $k < 2m$ so that for all nonnegative i , all boards of the form $G_{m,k+2mi}$ have outcome class V .*

Proof. Suppose this is false. Then for each k there is some i_k so that $G_{m,k+2mi_k}$ can be won by the horizontal player either going first or going second. Then the same is true for $G_{m,k+2mi}$ for any $i > i_k$ and only finitely many boards of height m are of outcome class V , a contradiction. \square

Proposition 17. *Let m and n be odd with $\gcd(m, n) = 1$. Then either the set of boards of height m or the set of boards of height n contains only finitely many boards of outcome class V .*

Proof. Assume that both sets contain infinitely many boards of outcome class V . By Lemma 16, for some k , all boards of form $G_{m,k+2mi}$ have outcome class V . The same is true for boards of the form $G_{mr,k+2mi}$ by the one-hand-tied principle.

Similarly (by diagonal reflection) all boards of the form $G_{\ell+2nj,ns}$ have outcome class H .

Since $\gcd(m, 2n) = 1 = \gcd(2m, n)$, there are choices of positive i, j, r , and s such that $mr = \ell + 2nj$ and $k + 2mi = ns$. Then the board $G_{mr,ns}$ has outcome class both H and V , a contradiction. \square

The same results hold for boards of widths six and eight with V replacing H for all heights greater than 31 by reflection across the diagonal.

5. Errata for Previous Work

5.1. Errata for Lachmann *et al.*

Lachmann *et al.* have a table of outcome classes similar to the table above. Their table contains a few errors.

In the table, they indicate which positions' outcome classes are calculated by brute force and which outcome classes follow from applying their rules. The outcome class of the 9×9 board does not follow from their rules as indicated in their table and must be calculated by brute force (the outcome class itself is correct).

On the other hand, the outcome classes for the 2×27 and 6×12 boards do not need to be calculated by brute force as indicated in their table but rather follow from their rules. The vertical player can win the 2×27 board going first by partitioning it into two 2×13 boards and since $27 = 13 + 14$, the one-hand-tied principle indicates that the horizontal player can also win the 2×27 board going first. The outcome class of the 6×12 board is calculated using their rules near the end of Section 3.

There is a more serious error in the transcription of the outcome class of the 4×13 board from [3]. The outcome class for that board is P (this was later verified by Bullock, who did not mention the discrepancy) but Lachmann *et al.* record it as V . This means that the outcome class of the 6×13 board must be checked by brute force (their entry is correct). It also means that their rules do not imply that the outcome class of the 13×17 board is N or H ; it may also be P . Furthermore, this implies by using their rules that the outcome class of the 4×21 board is H (this was later verified by Bullock).

5.2. Erratum for Bullock

In [5] (although not in [4]), Bullock records the outcome class of $G_{6,29}$ as NH ($1H$ in his notation). This turns out to be true, but it does not follow from any stated rule and this case is too large for a feasible verification with Obsequi.

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