



DIFFERENCES OF MULTIPLE FIBONACCI NUMBERS

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Abstract

We show that every integer can be written uniquely as a sum of Fibonacci numbers and their additive inverses, such that every two terms of the same sign differ in index by at least 4 and every two terms of different sign differ in index by at least 3. Furthermore, there is no way to use fewer terms to write a number as a sum of Fibonacci numbers and their additive inverses. This is an analogue of the Zeckendorf representation.

1. Main Results

The canonical way to write a natural number as a sum of Fibonacci numbers is the Zeckendorf representation, which is a sum of distinct nonconsecutive Fibonacci numbers [3]. Brown showed that every natural number can be represented uniquely as a sum of distinct Fibonacci numbers in which no two consecutive Fibonacci numbers smaller than the largest summand are omitted from the sum [1], and Bunder showed that every integer can be represented uniquely as a sum of distinct nonconsecutive Fibonacci numbers of negative index [2]. We present another way to uniquely write integers as sums, in which the summands are Fibonacci numbers and their additive inverses.

Define the Fibonacci *word length* of any integer $x \in \mathbb{Z}$ to be the least r such that $x = \sum_{i=1}^r a_i$, such that a_i or $-a_i$ is a Fibonacci number for each i . The Fibonacci word length of 0 is 0.

Index the Fibonacci numbers by $F_1 = 1, F_2 = 2$, so that the number 1 is unambiguously F_1 rather than F_0 . Let a *far-difference representation* be any sum of Fibonacci numbers and their additive inverses such that every two terms of the same sign are at least 4 apart in index, and every two terms of different sign are at least 3 apart in index.

Theorem 1. *Every $x \in \mathbb{Z}$ has a unique far-difference representation.*

Proof. First define $S(n)$ to be the quantity

$$S(n) = \sum_{0 < n-4i \leq n} F_{n-4i} = F_n + F_{n-4} + F_{n-8} + \cdots$$

For $n \leq 0$ we define $S(n)$ to be 0.

In any sum of Fibonacci numbers and their additive inverses, define the *main term* to be the term of greatest absolute value in the sum. Notice that if $x \in \mathbb{Z}$ has a far-difference representation with positive F_n as the main term, then $F_n - S(n - 3) \leq x \leq S(n)$. We claim that the positive integers are the disjoint union over all positive n of $[F_n - S(n - 3) .. S(n)]$. The first few such intervals are: $\{1\}, \{2\}, \{3\}, [4 .. 5], [6 .. 9], [10 .. 15], [16 .. 24], [25 .. 39], \dots$

Writing $S(n)$ as $F_n + S(n - 4)$ and using the easily-proven fact that $S(n - 4) + S(n - 2) = F_{n-1} - 1$, we find

$$\begin{aligned} (F_{n+1} - S(n - 2)) - S(n) &= (F_{n+1} - S(n - 2)) - (F_n + S(n - 4)) \\ &= F_{n+1} - F_n - (F_{n-1} - 1) = 1. \end{aligned}$$

Thus, if $x \in \mathbb{Z}$ is positive, there is a unique n such that $F_n - S(n - 3) \leq x \leq S(n)$. If x has a far-difference representation, then F_n must be the main term. Symmetrically, if x is negative, then there is a unique n such that $F_n - S(n - 3) \leq -x \leq S(n)$, and then $-F_n$ must be the main term of any far-difference representation of x . For the remainder of the proof, we assume without loss of generality that x is nonnegative.

We show by induction on n that if $0 \leq x \leq S(n)$, then x has a unique far-difference representation. As shown above, it is equivalent to show that out of the far-difference representations with main term F_k for $0 < k \leq n$, exactly one of these representations is equal to x . As the base case, we see that for $n = 0$, the unique far-difference representation of $x = 0$ is empty.

For the inductive step, suppose $S(n - 1) < x \leq S(n)$. Equivalently, $F_n - S(n - 3) \leq x \leq S(n)$. In the case $F_n - S(n - 3) \leq x \leq F_n$, we see $0 \leq F_n - x \leq S(n - 3)$, so by the inductive hypothesis $F_n - x$ has a unique far-difference representation with main term at most F_{n-3} . In the case $F_n \leq x \leq S(n)$, we see $0 \leq x - F_n \leq S(n - 4)$, so by the inductive hypothesis $x - F_n$ has a unique far-difference representation with main term at most F_{n-4} . In either case, adding F_n to the far-difference representation of $x - F_n$ gives the unique far-difference representation of x . \square

Theorem 2. *The number of terms in the far-difference representation of $x \in \mathbb{Z}$ is the Fibonacci word length of x .*

Proof. Let T be the set of ways to write $x \in \mathbb{Z}$ as a shortest sum of Fibonacci numbers and their additive inverses.

We define the following lexicographical ordering on T . For each sum in T , discard the signs so that what remains is a multiset of positive Fibonacci numbers, and we will order the multisets instead of the sums. Order these multisets first by greatest element. Then among multisets with the same greatest element, discard one copy of that greatest element; order the resulting multisets by greatest element, and so on, until the multisets are totally ordered. We claim that T is finite and that its greatest element in this order is a far-difference representation.

In order to show that T is a finite set, we show that the main term of a sum in T cannot be arbitrarily large. We show first that if T contains elements with arbitrarily large main terms, then it contains elements with arbitrarily large main terms in which all terms are distinct and more than 1 apart in index.

Every element of T has no two terms exactly 1 apart in index, because they could be replaced by a single term to make a shorter sum. Similarly, any element of T cannot have three of the same term, because those three terms could be replaced by two terms using the identity

$$F_n + F_n + F_n = F_{n-2} + F_{n+2}.$$

In any sum $N \in T$, let F_n be the greatest absolute value of any repeated term in N . Then we may use the identity

$$F_n + F_n = F_{n-2} + F_{n+1}$$

to replace the repeated F_n and find a different sum $N' \in T$. In N' , there is no repeated term of absolute value greater than or equal to F_n , because F_n and F_{n+1} cannot both appear as summands of N . The main term of N' is at least as large in absolute value as the main term of N , so we may continue to replace repeated terms to obtain an element of T in which all terms are distinct and more than 1 apart in index, in which the main term is at least as large in absolute value as the main term of N .

Suppose $N \in T$ has no repeated terms and has main term F_n . Then

$$x \geq F_n - \sum_{0 < n-2i < n} F_{n-2i}.$$

Again we have

$$\sum_{0 < n-2i < n} F_{n-2i} = F_{n-1} - 1,$$

so $x \geq F_n - (F_{n-1} - 1) = F_{n-2} + 1$. Because x is fixed, we conclude that T does not have elements with arbitrarily large main term, and that T is finite.

Because T is finite, there is a greatest element in the lexicographical ordering on T described at the beginning of the proof. We show that the greatest element $M \in T$ is a far-difference representation. First, in M no two terms of the same sign can be exactly 1 apart in index, and no two terms of different sign can be less than 3 apart in index, for otherwise M is not a shortest sum and cannot be in T :

$$F_n + F_{n+1} = F_{n+2},$$

$$F_n - F_n = 0,$$

$$F_{n+1} - F_n = F_{n-1},$$

$$F_{n+2} - F_n = F_{n+1}.$$

Then, we see that no two terms of the same sign can be 0, 2, or 3 apart in index, for if so we could replace them to make an element of T greater than M in our order:

$$\begin{aligned}F_n + F_n &= F_{n-2} + F_{n+1}, \\F_n + F_{n+2} &= F_{n+3} - F_{n-1}, \\F_n + F_{n+3} &= F_{n+4} - F_{n+1}.\end{aligned}$$

So, indeed, M is the far-difference representation of x . □

Theorem 3. *For every positive integer n , there exist infinitely many integers with Fibonacci word length n .*

Proof. For any nonnegative integer j , define

$$x_{j,n} = F_{4n+j} + \sum_{i=1}^{n-1} F_{4i}.$$

This sum is a far-difference representation of n terms, so the Fibonacci word length of $x_{j,n}$ is n . Then $\{x_{j,n}\}_{j=0}^{\infty}$ is a strictly increasing infinite sequence of integers each with Fibonacci word length n . □

For further research we would ask, with respect to a given sequence of natural numbers, are there integers with arbitrarily large word length? This question has been posed by Nathanson [4]. We would ask specifically about sequences satisfying linear recurrences; for which linear recurrences do the corresponding sequences produce arbitrarily large word length?

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