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THE SEQUENTIAL JOIN OF COMBINATORIAL GAMES

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Abstract

We study the sequential join of combinatorial games, and show that all combinatorial games form a monoid under this operation. We also show how this can be used to study games such as Childish Hackenbush and a new variant of normal play Hackenbush.

1. Introduction

A combinatorial game is a two player perfect information game, where there are no chance moves, e.g., rolling of dice, shuffling of cards, etc. A formal definition can be found in Winning Ways [2, Chapter 1]. Mathematically, we define combinatorial games themselves recursively following Conway [4].

Definition 1 *The options of a game G are the sets of all games that a player can move to from G and are denoted by:*

$G^L = \{$*All games that Left can move to from* $G\}$, *and*

$G^R = \{$*All games that Right can move to from* $G\}$

A game G is written as $\{G^L|G^R\}$, where G^L and G^R are the options of Left and Right respectively.

We abuse notation by letting G^L and G^R represent a set of options and the specific options themselves. The base case for the recursion is the game $\{|\}$, i.e. $G^L = G^R = \emptyset$. One thing that Conway noticed was that many games tend to break up into lots of smaller ones, which are played all at once and independently. That is, a move in one component will not affect any of the other components. This became the mathematical operator known as the disjunctive sum.

Definition 2 *The disjunctive sum of two games G and H is defined as follows:*

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$

What this operator[1] says is that when playing the game $G + H$, on your turn you may make a move in the game G or the game H but not both.

When studying combinatorial game theory, we generally look at games that naturally split up into the disjunctive sum of smaller components, e.g., Go (see [3]) or Hackenbush (see [2]). For this reason there is a lot known about the mathematical structure of games under the disjunctive sum, and more information about this operator can be found in Lessons in [1]-[4].

However there are many games, such as Childish Hackenbush [2], that do not naturally split up into the disjunctive sum of smaller components. Games such as these are played sequentially. That is, you must play each of the smaller components in a given order. This operator was first defined in Stromquist and Ullman [9], and as far as this author is aware, this is the only paper on the subject.

Definition 3 *The sequential join of two games G and H is defined as follows:*

$$G \rhd H = \begin{cases} \{G^L \rhd H | G^R \rhd H\}, \text{ if } G \neq \{|\} \\ H, \text{ Otherwise} \end{cases}$$

What this says is that when playing $G \rhd H$ you must first play in G, then H. Currently there is very little known about the structure of games under this operation, and since there are many games that can be effectively studied using this operator, it is worth looking at the mathematical structure of the operator in detail.

As has already been shown by Plambeck and Siegel [7], in their study of misère games, knowing that the underlying structure of any set is a monoid is extremely useful and gives a lot of power to the mathematician to help solve games of this type. Since we will show that all games form a monoid under the sequential join, this should prove to be extremely useful when studying any game that is played sequentially.

2. Structure of Combinatorial Games Under the Sequential Join

When studying a new type of summation the first step is to look at the addition tables for the outcome classes of games. All games fall into one of four outcome classes:

[1] We again abuse notation by allowing the comma to mean set union, and the "+" means take the disjunctive sum of the game with all games in the set, e.g. $G^L + H$ means take the disjunctive sum of H with all games in G^L.

1. Left win, denoted by $\mathcal{L} = \{G|\text{Left wins moving first and second in } G\}$.
2. Right win, denoted by $\mathcal{R} = \{G|\text{Right wins moving first and second in } G\}$.
3. Next player win, denoted by $\mathcal{N} = \{G|\text{The first player to move wins in } G\}$.
4. Previous player win, denoted by $\mathcal{P} = \{G|\text{The second player to move wins in } G\}$.

There are also two types of play in combinatorial game theory:

1. Normal play, where the last player to move is the winner.
2. Misère play, where the last player to move is the loser.

In general we tend to study normal play since this seems to be the more natural way to play a combinatorial game; however, it is still interesting to draw comparisons between normal play and misère play.

Theorem 4 *The addition table for the sequential join under misère rules is as follows:*

$G \rhd H$	$G \in \mathcal{N}$	$G \in \mathcal{P}$	$G \in \mathcal{L}$	$G \in \mathcal{R}$
$H \in \mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$
$H \in \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$
$H \in \mathcal{L}$	$\mathcal{L}, \mathcal{N}$	$\mathcal{L}, \mathcal{P}$	$\mathcal{L}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$
$H \in \mathcal{R}$	$\mathcal{R}, \mathcal{N}$	$\mathcal{R}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{R}$

Table 1: Outcome class of $G \rhd H$, under misère play

Proof.

Case 1: $G \in \mathcal{X}, H \in \mathcal{N} \Rightarrow G \rhd H \in \mathcal{X}$, where $\mathcal{X} = \mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{N}$.

Since these games are played in order G first then H, if a player has a winning strategy on G, then he has a winning strategy on $G \rhd H$. This is because he simply plays his winning strategy on G, forcing his opponent to move last in G, which means he can move first in H, and win, since H is a next player win.

Conversely if a player has no winning strategy in G, then he will have no winning strategy in $G \rhd H$. This is because his opponent can always force him to make the last move in G, and thus gain the first move in H and again since H is a next player win, his opponent will win.

Case 2: $G \in \mathcal{N}, H \in \mathcal{L} \Rightarrow G \rhd H \in \mathcal{L}$ or $\mathcal{N}$.

Left will always win moving first since Right can either eventually move to the game $\{|\}$, in which case the players will play H, which is a Left win. The only other alternative is that

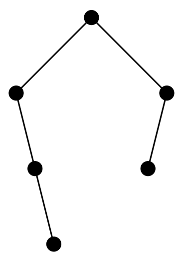
Right eventually moves to the game $\{|G''\}$ so Left will have no move in G, and therefore win.

Right moving first has Left moving eventually to $\{|\} \rhd H$ and so Right must play $H \in \mathcal{L}$ and so Left wins, or Left eventually moves to $\{G'|\} \rhd H$ and since Right will have no move, Right wins. So $G \rhd H$ is either an $\mathcal{L}$ or an $\mathcal{N}$ position.

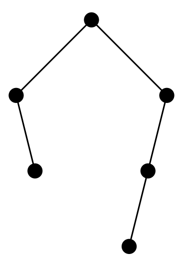
Taking $H \in \mathcal{R}$ will give a similar argument to the above case.

Case 3: $G \in \mathcal{N}, H \in \mathcal{P} \Rightarrow G \rhd H \in \mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$

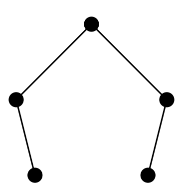
To prove this we need just four examples of games that are in each of the four outcome classes. They are as follows:



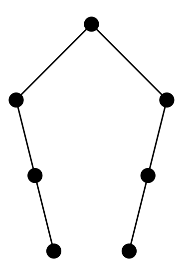
$\rhd H \in \mathcal{L}$



$\rhd H \in \mathcal{R}$



$\rhd H \in \mathcal{P}$



$\rhd H \in \mathcal{N}$

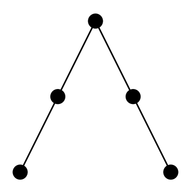
Case 4: $G, H \in \mathcal{L} \Rightarrow G \triangleright H \in \mathcal{L}$

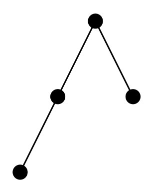
Since G is a Left win this means that Right moves last in G moving first or second. If Right eventually moves to the game $\{|\}$, then Left will play H, which is also a Left win. If Right eventually moves to the game $\{|H'\}$, then it will be Left's turn, and he will have no move and therefore win.

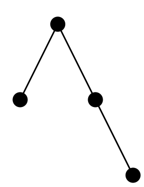
These are the only two possibilities for Right moving first or second, and so $G \triangleright H$ is a Left win.

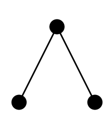
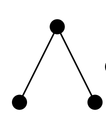
We can use an almost identical argument for $G, H \in \mathcal{R} \Rightarrow G \triangleright H \in \mathcal{R}$.

Case 5: $G, H \in \mathcal{P} \Rightarrow G \triangleright H \in \mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$. Consider the following examples:

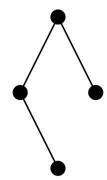
 $\triangleright H \in \mathcal{P}$

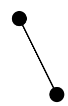
 $\triangleright H \in \mathcal{R}$

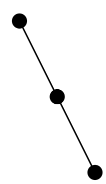
 $\triangleright H \in \mathcal{L}$

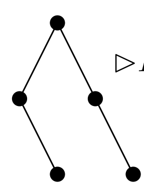
 $\triangleright$  $\in \mathcal{N}$

Case 6: $G \in \mathcal{L}, H \in \mathcal{P}$ or $\mathcal{R} \Rightarrow G \rhd H \in \mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$. Consider the following examples:


$\rhd H \in \mathcal{R}$


$\rhd H \in \mathcal{N}$


$\rhd H \in \mathcal{L}$


$\rhd H \in \mathcal{P}$

This case also implies that if $G \in \mathcal{R}$ then $G \rhd H \in \mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$, by replacing the $\mathcal{L}$ positions in the above for their negatives.

Case 7: $G \in \mathcal{P}, H \in \mathcal{L} \Rightarrow G \rhd H \in \mathcal{L}$ or $\mathcal{P}$

Right must always lose playing first, since he will either eventually move to the game $\{|\} \rhd H$ or to the game $\{|G'\} \rhd H$, and since $H \in \mathcal{L}$, Left will win both cases.

Left will only win playing first, if he eventually moves to the game $\{|\} \rhd H$, in which case this will be an $\mathcal{L}$ position. Otherwise he will eventually move to the game $\{G''|\} \rhd H$, and Right will win. Therefore this will be a $\mathcal{P}$ position.

Taking $H \in \mathcal{R}$ will give a similar argument to the above case.

This completes the proof of the table. $\square$

Theorem 5 *The addition table for the sequential join under normal play rules is as follows:*

$G \rhd H$	$G \in \mathcal{P}$	$G \in \mathcal{N}$	$G \in \mathcal{L}$	$G \in \mathcal{R}$
$H \in \mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{R}$
$H \in \mathcal{N}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$
$H \in \mathcal{L}$	$\mathcal{L}, \mathcal{P}$	$\mathcal{L}, \mathcal{N}$	$\mathcal{L}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$
$H \in \mathcal{R}$	$\mathcal{R}, \mathcal{P}$	$\mathcal{R}, \mathcal{N}$	$\mathcal{L}, \mathcal{R}, \mathcal{N}, \mathcal{P}$	$\mathcal{R}$

Table 2: Outcome class table of $G \rhd H$ under normal play

Proof. The proof of this is almost identical to the proof of Theorem 4. For the full proof see Stewart [8]. □

Definition 6 *$G \approx H$ means G has the same outcome as H*

Theorem 7 *All games, using misère and normal play rules, under the sequential join form a monoid.*

Proof. To prove this we need to show that an identity exists. Let $E_n = \{e | G \rhd e \approx e \rhd G \approx G$ for all normal play games $G\}$, $E_m = \{e | G \rhd e \approx e \rhd G \approx G$ for all misère play games $G\}$ and let $E = E_n \cup E_m$. The set E will act as the identity on the sets $\mathcal{L}, \mathcal{R}, \mathcal{P}$ and $\mathcal{N}$. We need to show that E is non-empty. To do that, consider the game $e = \{\{|\{|\}\}|\{\{|\}|\}\}$ as shown in Figure 1.

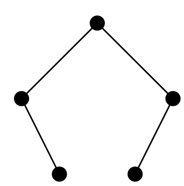


Figure 1: The game $e = \{\{|\{|\}\}|\{\{|\}|\}\}$

Since this game is a $\mathcal{P}$ in normal play and an $\mathcal{N}$ in misère, we know from the previous two theorems that $G \rhd e$ has the same outcome as G. So all that remains to show is that $e \rhd G$ has the same outcome as G.

If Left moves first in $e \rhd G$, then he must move to the game $\{|\{|\}\} \rhd G$, and therefore Right must move to the game $\{|\} \rhd G = G$, and therefore Left will move first in G, and by symmetry, Right moving first in $e \rhd G$ will also move first in G. For this reason playing $e \rhd G$ is equivalent to playing G under both misère and normal play rules, and therefore $e \rhd G \approx G$.

Thus $e \in E$, and so an identity exists. Hence, all games under both misère and normal play rules form a monoid under the sequential join. □

Corollary 8 *Under either normal or misère play rules, not all games are invertible under the sequential join.*

Proof. To find an inverse we must find a game G^{-1} such that $G \triangleright G^{-1} \in E$ and $G^{-1} \triangleright G \in E$, for all games G. So consider the game $G = \{\{\{|\}\}|\}$:

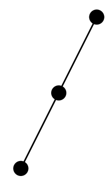


Figure 2: The game $G = \{\{\{|\}\}|\}$

If this game was invertible then $G \triangleright G^{-1} \triangleright Y$ will have the same outcome as Y for all games Y. Let Y be a $\mathcal{P}$ position under misère rules. Right will always win playing first since $G^R = \emptyset \Rightarrow (G \triangleright G^{-1} \triangleright Y)^R = \emptyset$, which means that this game cannot be a $\mathcal{P}$ position, under misère play rules.

If we let Y be an $\mathcal{N}$ position under normal play rules, then the same argument holds. Right will always lose playing first since $G^R = \emptyset \Rightarrow (G \triangleright G^{-1} \triangleright Y)^R = \emptyset$, which means that this game cannot be an $\mathcal{N}$ position under normal play rules. Therefore the game G is not invertible. This proves that not all games have an inverse. $\square$

3. Sequential Ordinary Hackenbush and Childish Hackenbush

Hackenbush is a game that is played on a graph, which is connected to a ground that is defined arbitrarily before the game begins, with coloured edges, where the players take turns to remove edges of their own colour. For the purposes of this paper we will only be considering the variant where the edges are only coloured red and blue. The rules of Ordinary Hackenbush are as follows:

1. The players take turns to remove edges.
2. Left removes blue edges, Right removes red edges.
3. Any edges not connected to the ground are also removed.
4. The last player to move wins.

3.1 Sequential Ordinary Hackenbush

To play Sequential Ordinary Hackenbush we add in the additional rule that you may not remove below certain *special* vertices in the graph that are defined before the game begins. These vertices will be marked black, while normal vertices will be white. An example of a game position is shown in Figure 3.

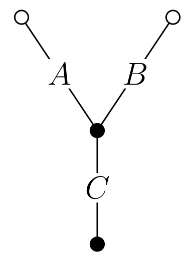


Figure 3: An example of a Sequential Ordinary Hackenbush position

In this game players must play the strings or edges A and B before C. As a summation it would be written as $(A + B) \rhd C$, since we must play the game $A + B$ before C.

A more complicated example is depicted in Figure 4. The corresponding sum is:

$$(((((I + J) \rhd H) + E + D) \rhd B) + (F + G) \rhd C) \rhd A$$

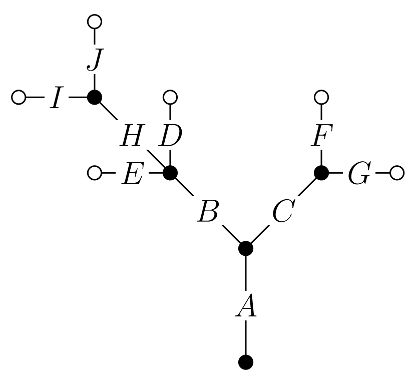


Figure 4: A more complicated example of Sequential Ordinary Hackenbush

Again each edge in the graph could represent an Ordinary Hackenbush string, or a general Ordinary Hackenbush position. By using what we know about disjunctive sums and combining it with the tables from the previous section, it is possible in some cases to work out what the outcome class is. For example, if $A, B \in \mathcal{L}$ and $C \in \mathcal{P}$ then $(A + B) \rhd C \in \mathcal{L}$. However, if $A \in \mathcal{L}$, $B \in \mathcal{L}$ and $C \in \mathcal{N}$, then $(A+B) \rhd C \in \mathcal{L}, \mathcal{R}, \mathcal{P}, \mathcal{N}$. In this case it is not possible to know what the outcome class is, and so a deeper understanding of the sequential join operator is needed.

3.2 Childish Hackenbush

Childish Hackenbush has the same rules as Ordinary Hackenbush, but we add in the extra rule that a player may not remove any edge that disconnects another edge from the ground. This game is in fact played entirely sequentially, and we can obtain the same sums as the previous section by replacing the strings in Figures 3 and 4 with Childish Hackenbush strings.

4. Conjectures and Open Problems

My main conjecture is the following:

Conjecture 9 *$E_n \equiv E_m$, i.e. they contain exactly the same games.*

However there is a problem that goes along with that:

Problem 10 *Which games are in the sets E_n and E_m?*

We have shown that they are non-empty and contain at least 1 element, but it is still completely unclear to us exactly which games belong in the sets.

Problem 11 *For any three outcome classes A, B and C, is it possible to classify which games $G \in A$ and $H \in B$ so that $G \triangleright H \in C$?*

This relates more specifically to the problem of Childish Hackenbush and other related sequential games, since having this classification would allow us to solve this game completely.

Acknowledgements

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References

[1] M. Albert, R. Nowakowski, D. Wolfe, *Lessons in Play*, A.K. Peters 2007.

[2] E. Berlekamp, J. Conway, R. Guy, *Winning Ways for your Mathematical Plays*, Volumes 1-4, A.K. Peters 2002.

[3] E. Berlekamp, D. Wolfe, *Mathematical Go: Chilling Gets the Last Point*, A.K. Peters 1994.

[4] J. Conway, *On Numbers and Games*, A.K. Peters 2000.

[5] P. Grundy, C. Smith *Disjunctive games with the last player losing*, Proc. Camb. Philos, Soc., **52**(1956), 443-458.

[6] T. Plambeck, *Taming the Wild in Impartial Combinatorial Games*, Integers, Volume 5(1)(2005)

[7] T. Plambeck, A. Siegel, *Misère Quotients for Impartial Games*, http://www.integraldomain.net/aaron/

[8] F. Stewart, *Finding Structure in Misère Games and Studying Variants of Misère Hackenbush*, MSc Thesis, Dalhousie University, Department of Mathematics and Statistics, 2006.

[9] W. Stromquist, D. Ullman, *Sequential Join of a Combinatorial Game*, Journal of Theoretical Computer Science, Vol.119 Number 2, Pg: 311-321, 1993