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COMBINED ALGEBRAIC PROPERTIES OF CENTRAL* SETS

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Abstract

In this work we prove that in the semigroup $(\mathbb{N},+)$ if $\langle x_n\rangle_{n=1}^\infty$ is a sequence such that $FS(\langle x_n\rangle_{n=1}^\infty)$ is piecewise syndetic, then for any central* set A there exists a sum subsystem $\langle y_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ with the property that $FS(\langle y_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)\subseteq A$.

1. Introduction

Given any discrete semigroup $(S,\cdot)$, βS is the Stone-Čech compactification of S and the operation $\cdot$ on S has a natural extension to βS making βS a compact right topological semigroup with S contained in its topological center. (By "right topological" we mean that for each $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ is continuous, where $\rho_p(q)=q\cdot p$. By the "topological center" we mean the set of points p such that λ_p is continuous, where $\lambda_p(q)=p\cdot q$.)

As a compact right topological semigroup, βS has a smallest two sided ideal denoted by $K(\beta S)$. Further, $K(\beta S)$ is the union of all minimal right ideals of βS and is also the union of all minimal left ideals. (See [5], Chapter 2 for these and any other unfamiliar facts about compact right topological semigroups.) Any compact right topological semigroup has an idempotent and one can define a partial ordering of the idempotents by $p\le q$ if and only if $p=p\cdot q=q\cdot p$. An idempotent p is "minimal" if and only if p is minimal with respect to the order $\le$. Equivalently, an idempotent p is minimal if and only if $p\in K(\beta S)$.

The algebraic structure of the smallest ideal of βS has played a significant role in Ramsey Theory. For example, a subset A of $(S,\cdot)$ is defined to be central if it is a member of an idempotent in $K(S)$. It is known that any central subset of $(\mathbb{N},+)$ is guaranteed to have substantial additive structure. But Theorem 16.27 of [5] shows that central sets in $(\mathbb{N},+)$ need have no multiplicative structure at all. On the other hand, in [2] we see that sets

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which belong to every minimal idempotent of $\mathbb{N}$, called central* sets, must have significant multiplicative structure. In fact central* sets in any semigroup $(S, \cdot)$ are defined to be those sets which meet every every central set.

We now present three results that will be useful in this article. Theorem 1.1 is in [5] as Corollary 16.21, Theorem 1.2 is in [2] as Theorem 2.6, and Theorem 1.3 is in [4] as Theorem 2.11.

Theorem 1.1. *If A is a central* set in $(\mathbb{N}, +)$ then it is central in $(\mathbb{N}, \cdot)$.*

In [5], it is also proved that IP* sets in $(\mathbb{N}, +)$ are guaranteed to have substantial combined additive and multiplicative structure, where a set $A \subseteq \mathbb{N}$ is called an IP* set if it belongs to every idempotent in $\mathbb{N}$. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, we denote $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, where for any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X, and $FP(\langle x_n \rangle_{n=1}^{\infty})$ is the product analogue of the above. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, we say that $\langle y_n \rangle_{n=1}^{\infty}$ is a *sum subsystem* of $\langle x_n \rangle_{n=1}^{\infty}$ provided there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ of nonempty finite subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ and $y_n = \sum_{t \in H_n} x_t$ for each $n \in \mathbb{N}$.

Theorem 1.2. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence and A be an IP* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$*

A strongly negative answer to the partition analogue of the above result is presented in [4]. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, $PS(\langle x_n \rangle_{n=1}^{\infty}) = \{x_m + x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$ and $PP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_m \cdot x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$.

Theorem 1.3. *There exists a finite partition $\mathcal{R}$ of $\mathbb{N}$ with no one-to-one sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $PS(\langle x_n \rangle_{n=1}^{\infty}) \cup PP(\langle x_n \rangle_{n=1}^{\infty})$ is contained in one cell of the partition $\mathcal{R}$.*

The main aim of this article is to show that central* sets also possess some IP* set-like properties for some specified sequences.

2. The Proof of the Main Theorem

We first introduce the following notion for our purpose.

Definition 2.1. Let $(S, \cdot)$ be a commutative semigroup. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S is said to be a *minimal* sequence if $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})} \bigcap K(\beta S) \neq \emptyset$.

It is already known that $\langle 2^n \rangle_{n=1}^{\infty}$ is a minimal sequence while the sequence $\langle 2^{2n} \rangle_{n=1}^{\infty}$ is not a minimal sequence. In [1] it is proved that in the semigroup $(\mathbb{N}, +)$ minimal sequences are nothing but those for which the set $FS\langle x_n \rangle_{n=1}^{\infty}$ is large enough, i.e., it meets the smallest ideal $K(\beta\mathbb{N})$ of $(\beta\mathbb{N}, +)$.

Lemma 2.2. *If A is a central set in $(\mathbb{N},+)$ then nA is also central for any $n \in \mathbb{N}$.*

Proof. [3], Lemma 3.8. □

Given $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, $n^{-1}A = \{m \in \mathbb{N} : nm \in A\}$ and $-n+A = \{m \in \mathbb{N} : n+m \in A\}$.

Lemma 2.3. *If A is a central* set in $(\mathbb{N},+)$ then $n^{-1}A$ is also central* for any $n \in \mathbb{N}$.*

Proof. Let A be a central* set and $t \in \mathbb{N}$. To prove that $t^{-1}A$ is a central* set it is sufficient to show that for any central set C, $C \cap t^{-1}A \neq \emptyset$. Since C is central tC is also central so that $A \cap tC \neq \emptyset$. Choose $n \in tC \cap A$ and $k \in C$ such that $n = tk$. Therefore $k = n/t \in t^{-1}A$ so that $C \cap t^{-1}A \neq \emptyset$. □

We now show that all central* sets have a substantial multiplicative property.

Theorem 2.4. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a minimal sequence and A be a central* set in $(\mathbb{N},+)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Since $\langle x_n \rangle_{n=1}^{\infty}$ is a minimal sequence in $\mathbb{N}$ we can find some minimal idempotent $p \in \mathbb{N}$ for which $FS(\langle x_n \rangle_{n=1}^{\infty}) \in p$. Again, since A is a central* subset of $\mathbb{N}$, by the previous lemma for every $n \in \mathbb{N}$, $n^{-1}A \in p$. Let $A^* = \{n \in A : -n + A \in p\}$. Then by ([5], Lemma 4.14) $A^* \in p$. We can choose $y_1 \in A^* \cap FS(\langle x_n \rangle_{n=1}^{\infty})$. Inductively let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^{m}$, $\langle H_i \rangle_{i=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following properties:

1. $i \in \{1, 2, \cdots, m-1\}$ $\max H_i < \min H_{i+1}$;
2. If $y_i = \Sigma_{t \in H_i} x_t$ then $\Sigma_{t \in H_{m+1}} x_t \in A^*$ and $FP(\langle y_i \rangle_{i=1}^{m}) \subseteq A$.

We observe that $\{\Sigma_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\} \in p$. It follows that we can choose $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$, $\Sigma_{t \in H_{m+1}} x_t \in A^*$, $\Sigma_{t \in H_{m+1}} x_t \in -n + A^*$ for every $n \in FS(\langle y_i \rangle_{i=1}^{m})$ and $\Sigma_{t \in H_{m+1}} x_t \in n^{-1}A^*$ for every $n \in FP(\langle y_i \rangle_{i=1}^{m})$. Putting $y_{m+1} = \Sigma_{t \in H_{m+1}} x_t$ shows that the induction can be continued and proves the theorem. □

Notice that if A is not an IP*-set, then there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \cap A = \emptyset$ so Theorem 1.2 in fact characterizes IP* sets. We do not know whether Theorem 2.6 similarly characterizes central* sets.

Question 2.5. *Given a non-central* set A in $(\mathbb{N},+)$, can we find a minimal sequence $\langle y_n \rangle_{n=1}^{\infty}$ such that for no sum subsystem $\langle x_n \rangle_{n=1}^{\infty}$ does one have $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

In [1] a notion of sequence named *nice* sequence has been introduced. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $(\mathbb{N}, +)$ is called a *nice* sequence if it satisfies the uniqueness of finite products and for all $m \in \mathbb{N} \setminus FS(\langle x_n \rangle_{n=1}^{\infty})$ there is some $k \in \mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \cap (m + FS(\langle x_n \rangle_{n=k}^{\infty})) = \emptyset$, where $\langle x_n \rangle_{n=1}^{\infty}$ is said to satisfy uniqueness of finite products provided that if $F, G \in \mathcal{P}_f(\mathbb{N})$ and $\sum_{k \in F} x_k = \sum_{k \in G} x_k$, one must have $F = G$. The following theorem follows from Corollary 4.2 of [1].

Theorem 2.6. *If $\langle x_n \rangle_{n=1}^{\infty}$ is a nice minimal sequence in $(\mathbb{N}, +)$ then we have that $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic for each $m \in \mathbb{N}$.*

In the following theorem we provide a partial answer to the above question by producing a non-central* set for which every nice minimal sequence satisfies the conclusion of Theorem 2.4. The author thanks Prof. Neil Hindman for providing the proof of this theorem.

Theorem 2.7. *Let $A = \bigcup_{n=1}^{\infty} \{2^{2n}, 2^{2n} + 1, \ldots, 2^{2n+1} - 1\}$ and $\langle x_n \rangle_{n=1}^{\infty}$ be a nice minimal sequence in $\mathbb{N}$. Then there is a sum subsystem $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subset A$.*

Proof. By Theorem 2.6, we have $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic for each $m \in \mathbb{N}$. We inductively construct sequences $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ and $\langle k_n \rangle_{n=1}^{\infty}$ of integers such that for each $n \in \mathbb{N}$,

(a) $\max H_n < \min H_{n+1}$,

(b) $2^{2k_{n+1}+1} - 2^{2k_{n+1}+1/2} > \sum_{r=1} \sum_{t \in H_r} x_t$,

(c) $2^{2k_n} < \sum_{t \in H_n} x_t < 2^{2k_n + 1/2^n}$.

Having chosen these terms through n, let $m = \max H_n + 1$ and pick b such that the gaps of $FS(\langle x_n \rangle_{n=m}^{\infty})$ are bounded by b. Then pick k_{n+1} satisfying (b) such that $2^{2k_{n+1}+1/2^{n+1}} - 2^{2k_{n+1}} > b$. Then pick H_{n+1} in $\mathcal{P}_f(\mathbb{N})$ with $\min H_{n+1} \geq m$ such that $2^{2k_{n+1}} < \sum_{t \in H_{n+1}} x_t < 2^{2k_{n+1}+1/2^n} + b$. Thus the induction is complete.

Now we take $y_n = \sum_{t \in H_n}$. Then $\langle y_n \rangle_{n=1}^{\infty}$ becomes a sum subsystem of $\langle x_n \rangle_{n=1}^{\infty}$. Now if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \max F$ then clearly $2^{2k_n} \leq \sum_{t \in F} y_t \leq \sum_{t=1}^{m} y_m \leq 2^{2k_{n+1}+1} - 1$, so that $FS(\langle y_n \rangle_{n=1}^{\infty}) \subset A$. Again if $G \in \mathcal{P}_f(\mathbb{N})$ from (c) it follows easily that $2^{2 \quad_{t \in G} k_m} \leq \prod_{t \in G} y_t < 2^{2 \quad_{t \in G} k_t + 1}$ and hence $FP(\langle y_n \rangle_{n=1}^{\infty}) \subset A$. □

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