k-FIXED-POINTS-PERMUTATIONS

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Abstract

In this paper we study the *k-fixed-points* statistic over the symmetric group. We will give some combinatorial interpretations to the relations defining them as well as their generating functions. A combinatorial interpretation directly on derangements of the famous relation on derangement numbers $d_n = nd_{n-1} + (-1)^n$ will be given.

1. Introduction

Euler (see [1] and [4]) introduced the difference table $(e_n^k)_{0\leq k\leq n}$, where e_n^k are defined by

$$e_n^n = n! \text{ and } e_n^{k-1} = e_n^k - e_{n-1}^{k-1} \text{ for } 1 \leq k \leq n,$$

without giving their combinatorial interpretation. In our previous paper [11], we studied these numbers, which generalize the derangement theory, through the study of k-successions. The first values of the numbers e_n^k are given in the following table:

e_n^k						
	$k=0$	1	2	3	4	5
$n=0$	0!					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

and their generating functions are defined by

$$\begin{cases} E^{(k)}(u) = \sum_{n\geq 0} e^k_{n+k} \frac{u^n}{n!} = k! \frac{\exp(-u)}{(1-u)^{k+1}} \\ E(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} e^k_{n+k} \frac{x^k}{k!}\frac{u^n}{n!} = \frac{\exp(-u)}{1-x-u}. \end{cases}$$

The motivation of this paper is to study the numbers d^k_n which are obtained from the numbers e^k_n by dividing them by $k!$. It follows straightforwardly that their generating functions are defined by

$$\begin{cases} D^{(k)}(u) = \sum_{n\geq 0} d^k_{n+k} \frac{u^n}{n!} = \frac{\exp(-u)}{(1-u)^{k+1}} \\ D(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} d^k_{n+k} x^k \frac{u^n}{n!} = \frac{\exp(-u)}{1-x-u}. \end{cases}$$

We then obtain the following table for some first values of the numbers d^k_n:

d^k_n						
	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	1	1	1			
3	2	3	2	1		
4	9	11	7	3	1	
5	44	53	32	13	4	1

.

By a simple computation, we can find that the numbers d^k_n satisfy the following recurrences:

$$\begin{cases} d^k_k = 1 \\ d^k_n = (n-1)d^k_{n-1} + (n-k-1)d^k_{n-2} \text{ for } n > k \geq 0. \end{cases}$$

The aims of this paper are to give combinatorial interpretations of these numbers. We will give a combinatorial bijection to the unexpected relation

$$d^k_n + d^{k-1}_{n-2} = n d^k_{n-1}$$

which is a generalization of the famous recurrence on derangement numbers (see, e.g., [2], [5], [14]):

$$d_n = n d_{n-1} + (-1)^n.$$

The derangement case corresponds to $k = 0$, if we set

$$d^{-1}_{-1} = 1 \text{ and } d^{-1}_{n-1} + d^{-1}_n = 0 d^0_n,$$

that is, $d^{-1}_{n-1} + d^{-1}_n = 0$, then we obtain

$$d^{-1}_n = (-1)^{n+1}.$$

Désarmenien [2], Remmel [12] and Wilf [16] each gave a combinatorial proof of this last relation with other objects which are in bijection with derangements, but never directly on derangements. Many authors (see, e.g., [3], [6], [7], [8], [9], [10], [15]) have studied in depth the numbers d_n. A bijective proof directly over derangements, or permutations without fixed points, for this last relation of derangement numbers will be given in a separate section. Let us denote by $[n]$ the interval $\{1, 2, \cdots, n\}$, and by σ a permutation of the symmetric group $\mathfrak{S}_n$. In this paper, we will use the linear notation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, as well as the notation of the decomposition into a product of disjoint cycles, to represent a permutation.

Definition 1.1. We say that an integer i is a *fixed point* of a permutation σ if $\sigma(i) = i$. We will denote by $\mathrm{Fix}(\sigma)$ the set of fixed points of the permutation σ.

Definition 1.2. We say that a permutation σ is a *k-fixed-points-permutation* if for all integers i in the interval $[k]$, $\sigma^p(i) \notin [k] \setminus \{i\}$ for all integers p and $\mathrm{Fix}(\sigma) \subseteq [k]$.

We will denote by D_n^k the set of k-fixed-points-permutations of the symmetric group $\mathfrak{S}_n$.

Example 1.3. We have

$$D_1^0 = \{\},\ D_1^1 = \{1\},$$

$$D_2^0 = D_2^1 = \{21\},\ D_2^2 = \{12\}.$$

$$D_3^0 = \{231, 312\},\ D_3^1 = \{132, 231, 312\},\ D_3^2 = \{132,\ 312\},\ D_3^3 = \{123\}$$

Remark 1.4. The permutation $12\cdots k$ is the only k-fixed-points-permutation of the symmetric group $\mathfrak{S}_k$.

2. Numbers d_n^k

2.1. First Relation for the Numbers d_n^k

Theorem 2.1. *For* $0 \leq k \leq n-1$, *we have*

$$d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k.$$

To prove this theorem, let us consider the following definition.

Definition 2.2. Let the map $\varphi : D_n^k \to [n-1] \times D_{n-1}^k \cup [n-k-1] \times D_{n-2}^k$, which associates to each permutation σ a pair $(m, \sigma') = \varphi(\sigma)$, be defined as follows:

1. If the integer n is in a cycle of length greater than or equal to 3, or the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) \leq k$, then the integer m is equal to $\sigma^{-1}(n)$ and the permutation σ' is obtained from the permutation σ by removing the integer n from his cycle. (Note that the permutation σ' is indeed an element of the set D^k_{n-1}.)

2. If the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) > k$, then the integer m is equal to $\sigma(n)$ and the permutation σ' is obtained from the permutation σ by removing the cycle $(\sigma(n), n)$ and then decreasing by 1 all integers between $\sigma(n)+1$ and $n-1$ in each cycle. (Note that the permutation σ' is indeed an element of the set D^k_{n-2}.)

Remark 2.3. If the integer n is greater than k and $\sigma \in D^k_n$, then $\sigma'(n) \neq n$.

Proposition 2.4. *The map φ is bijective.*

Proof. Notice that a pair (m, σ') in the image $\varphi(D^k_n)$ is contained either in the set of all pairs of $[n-1] \times D^k_{n-1}$ if the integer n lies in a cycle of length greater than 2 or equal to 2 and $\sigma'(n) \leq k$, or in the set of all pairs of $[n-k-1] \times D^k_{n-2}$ if the integer n lies in a cycle of length equal to 2 and $\sigma'(n) > k$. Define a map $\tilde{\varphi} : [n-1] \times D^k_{n-1} \cup [n-k-1] \times D^k_{n-2} \to D^k_n$ so that the permutation $\sigma' = \tilde{\varphi}(m, \sigma)$ is obtained as follows:

- either by inserting the integer n in a cycle of the permutation σ after the integer $m \in [n-1]$ if σ is an element of the set D^k_{n-1}. In such case, the integer n lies in a cycle of length greater to 2 or in a transposition and $\sigma(n) \leq k$.

- or by creating the transposition (m, n) with $k < m \leq n-2$ and then increasing by 1 all integers between m and $n-2$ in each cycle of the permutation σ if the permutation σ is an element of the set D^k_{n-2}. In such case, the integer n is in a transposition and $\sigma(n) > k$.

The map $\tilde{\varphi}$ is the inverse of the map φ. □

Corollary 2.5. *The number d^k_n equals the cardinality of the set of k-fixed-points-permutations in the symmetric group $\mathfrak{S}_n$.*

Proposition 2.6. *For all integers k, we have $d^k_k = 1$.*

Proof. The permutation $12\ldots k$ is the only k-fixed-points permutation of the symmetric group $\mathfrak{S}_k$. □

2.2. Second Relation for the Numbers d_n^k

Another relation satisfied by the numbers d_n^k can be easily deduced from the generating function, but we will give its combinatorial interpretation.

Definition 2.7. Let the map $\vartheta : D_{n-1}^{k-1} \cup D_n^{k-1} \to [k] \times D_n^k$, which associates to a permutation σ a pair $(m, \sigma') = \vartheta(\sigma)$, be defined as below:

1. If $\sigma \in D_{n-1}^{k-1}$, then the integer m is equal to k and the permutation σ' is obtained from the permutation σ by creating the cycle (k) and then increasing by 1 all integers greater than or equal to k in each cycle of the permutation σ.
2. If $\sigma \in D_n^{k-1}$, then the integer m is equal to the smallest integer in the cycle that contains the integer k, and the permutation σ' is obtained from the permutation σ by removing the word $k\sigma(k)\cdots\sigma^{-1}(m)$ from that cycle and then creating the cycle $(k\sigma(k)\cdots\sigma^{-1}(m))$.

Proposition 2.8. *The map ϑ is a bijection.*

Proof. The map ϑ is injective. It suffices to show that ϑ is surjective. Let us look at various cases of the pair (m, σ').

1. If $m = k$ and $\sigma'(k) = k$, then we define the permutation σ by deleting the cycle (k) and then decreasing by 1 all integers greater than k in each cycle. It follows straightforwardly that the permutation σ is an element of the set D_{n-1}^{k-1}.
2. If $m = k$ and $\sigma'(k) \neq k$, then $\sigma = \sigma'$ and $\sigma \in D_n^{k-1}$.
3. If $m \neq k$, then the permutation σ is obtained from the permutation σ' by removing the cycle which contains k and then inserting the word $k\sigma'(k)\sigma'^2(k)\cdots$ in the cycle which contains the integer m just before the integer $\sigma'^{-1}(m)$. The permutation σ is indeed an element of the set D_n^{k-1}.

It is impossible by construction of the map ϑ that $m = k$ and the integer k is in the same cycle as an integer smaller than k. □

Theorem 2.9. *For all integers $1 \leq k \leq n$, we have*

$$kd_n^k = d_{n-1}^{k-1} + d_n^{k-1}.$$

Proof. By the bijection ϑ, we have

$$\#D_{n-1}^{k-1} + \#D_n^{k-1} = \#[k] \times D_n^k,$$

that is,

$$kd_n^k = d_{n-1}^{k-1} + d_n^{k-1}.$$

□

2.3. Third Relation for the Numbers d_n^k

The following unexpected relation is a generalization of the famous relation on derangement numbers and we will give a bijective proof of it.

Theorem 2.10. *For all integers* $0 \leq k \leq n-1$, *one has*

$$nd_{n-1}^k = d_n^k + d_{n-2}^{k-1}.$$

Proof. Let us consider the map $\varsigma : [n] \times D_{n-1}^k \longrightarrow D_n^k \cup D_{n-2}^{k-1}$, which associates to a pair (m, σ) a permutation $\sigma' = \varsigma((m, \sigma))$, defined in the following way:

1. If $m < n$, then the permutation σ' is obtained from the permutation σ by inserting the integer n in the cycle which contains m just before the integer m itself. The permutation σ' is indeed an element of the set D_n^k.
2. If $m = n$ and $\sigma(1) \neq 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the integer $\sigma(1)$ and then creating the cycle $(n \quad \sigma(1))$. The permutation σ' is indeed an element of the set D_n^k and $\sigma'(n) > k$.
3. If $m = n$ and $\sigma(1) = 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the cycle (1) and then by decreasing by 1 all integers in each cycle. It follows straightforwardly that the permutation σ' is an element of the set D_{n-2}^k.

It is clear that the map ς is injective. Hence, to show it is bijective, it suffices to show that ς is surjective. Let us look at the various cases of the permutation σ'.

1. If the permutation σ' is an element of the set D_n^k and the cycle that contains n is different from the transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the pair (m, σ) is defined by $m = \sigma'^{-1}(n)$, and the permutation σ is obtained by removing the integer n from the cycle containing it.
2. If the permutation σ' is an element of the set D_n^k and the c ycle that contains n is a transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the pair (m, σ) is defined by $m = n$ and the permutation σ is obtained by removing the cycle $(n \quad \sigma'(n))$ and inserting the integer $\sigma'(n)$ in the cycle that contains the integer 1 just after 1.
3. If the permutation σ' is an element of the set D_{n-2}^{k-1}, then the pair (m, σ) is defined by $m = n$ and the permutation σ is obtained by increasing by 1 all the integers in each cycle of the permutation σ' and then creating the new cycle (1).

□

Remark 2.11. Theorems 2.1 and 2.9 together imply Theorem 2.10 as follows. Let

$$F(n,k) = nd_{n-1}^k - d_n^k - d_{n-2}^{k-1}$$

$$G(n,k) = kd_n^k - d_{n-1}^{k-1} - d_n^{k-1}.$$

Then the identity in Theorem 2.1 can be rewritten as

$$F(n,k) + F(n-1,k) = G(n-2,k).$$

So, since $G(n,k) = 0$ for all $n \geq k \geq 0$, by Theorem 2.9, we get

$$F(n,k) = (-1)^{n-k-1} F_{k+1}^k = 0 \text{ (from Theorem 2.1) for all } n \geq k \geq 0.$$

It seems worth considering whether or not the sieve method can also be generalized using the above relation between F and G.

3. The Famous $d_n = nd_{n-1} + (-1)^n$

Notice that the set D_n of derangements or permutations without fixed points is equal to the set D_n^0.

Definition 3.1. Let us define the *critical derangement* $\Delta_n = (1\,2)(3\,4)\cdots(n-1 \quad n)$ if the integer n is even, and the sets

- $E_n = \{\Delta_n\}$ if the integer n is even, and $E_n = \emptyset$ otherwise,
- $F_n = \{(n, \Delta_{n-1})\}$ if the integer n is odd, and $F_n = \emptyset$ otherwise.

Let $\tau : [n] \times D_{n-1} \setminus F_n \to D_n \setminus E_n$ be the map which associates to a pair (i, δ) a permutation $\delta' = \tau((i, \delta))$ defined as follows:

1. If the integer $i < n$, then the permutation $\delta' = \delta(i \quad n)$. In other words, the permutation δ' is obtained from the permutation δ by inserting the integer n in the cycle that contains the integer i just after the integer i.
2. If the integer $i = n$, then let p be the smallest integer such that the transpositions $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the permutation δ and the transposition $(2p+1 \quad 2p+2)$ is not.
   (a) If $\delta(2p+1) = 2p+2$, then the permutation δ' is obtained from the permutation δ by removing the integer $2p+1$ from the cycle that contains it, and then creating the new cycle $(2p+1 \quad n)$.

(b) If $\delta(2p+1) \neq 2p+2$, then we have to distinguish the following two cases:

i. If the length of the cycle that contains the integer $2p+1$ is equal to 2, then the permutation δ' is obtained from the permutation δ by removing the cycle $(2p+1\, \delta(2p+1))$, and then inserting the integer $2p+1$ in the cycle that contains the integer $2p+2$ just before the integer $2p+2$ and creating the new cycle $(\delta(2p+1) \quad n)$.

ii. If the length of the cycle that contains the integer $2p+1$ is greater than 2, then the permutation δ' is obtained from the permutation δ by removing the integer $\delta(2p+1)$ and then creating the new cycle $(\delta(2p+1) \quad n)$.

Proposition 3.2. *The map τ is bijective.*

Proof. Notice that the only pair (i, δ) which is not defined by the map τ is the pair (n, Δ_{n-1}) if the integer $n-1$ is even. Notice also that the image $\tau([n-1] \times D_{n-1})$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length greater than or equal to 3, and the image $\tau(\{n\} \times D_{n-1} \setminus F_n)$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length 2. So we need only show that there exists a map ζ such that

- associates an element of $[n-1] \times D_{n-1}$ with every derangement of D_n in which the integer n lies in a cycle of length greater or equal to 3.

- associates an element of $\{n\} \times D_{n-1} \setminus F_n$ with every derangement of D_n in which the integer n lies in a cycle of length 2.

- is the inverse of τ.

It is straightforward to verify that the map ζ is defined as follows:

1. If the integer n lies in a cycle of length greater or equal to 3, then $\zeta(\delta)$ is the pair (i, δ') where $i = \delta^{-1}(n)$, and the permutation δ' is obtained by removing the integer n from the derangement δ. The permutation δ' is a derangement of D_{n-1} and the integer i is smaller than n.

2. If the integer n lies in a cycle of length 2, then let p the smallest nonnegative integer such that $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the derangement δ while the transposition $(2p+1 \quad 2p+2)$ is not.

   (a) If $\delta(n) = 2p+1$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad 2p+1)$ and then inserting the integer $2p+1$ in the cycle which contains the integer $2p+2$ just before the integer $2p+2$. In other words, we have $\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad n)(2p+2\ldots)\cdots$ and $\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots$.

(b) If $\delta(2p+1) \neq n$, then we have to distinguish the following two cases:

i. If $\delta(2p+1) \neq 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n\,\delta(n))$ and then inserting the integer $\delta(n)$ in the cycle which contains the integer $2p+1$ just before the integer $2p+1$. In other words, we have $\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1\ldots)\cdots(\delta(n) \quad n)\cdots$ and $\delta' = (12)(34)\cdots\ (2p-1 \quad 2p)(2p+1\ldots\delta(n))\cdots$.

ii. If $\delta(2p+1) = 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad \delta(n))$ and the integer $2p+1$ and then creating the new cycle $(2p+1 \quad \delta(n))$. In other words, we have $\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots(\delta(n) \quad n)\cdots$ and $\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad \delta(n))(2p+2\ldots)\cdots$.

Notice that the derangement Δ_n, if the integer n is even, is the only derangement which is not defined by the map ζ. □

Corollary 3.3. *If the integer n is even, then we have $d_n = nd_{n-1} + 1$. If the integer n is odd, then we have $d_n + 1 = nd_{n-1}$.*

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