On Subordination and Superordination of New Multiplier Transformation

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Abstract
Let \( \phi_\mu(z, m, c) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+c)^m}, \quad z \in \mathbb{C}, |z| < 1, c \in \mathbb{C} / \{0, -1, -2, \ldots\}, \mu, m \in \mathbb{C} \)
be the generalized Hurwitz–Lerch Zeta function. We consider

\[
\phi(z, m, c) = \sum_{k=0}^{\infty} \frac{(1+c)^m}{k} \frac{z^k}{k+c} = \left[ \frac{\log_{1+c}(1) + \log_{1+c}(1+c)}{1+c} \right] \frac{z^k}{k+c},
\]

and define

\[
[zF(z, m, c)] * [zF(z, m, c)]^\lambda = \frac{z}{(1-z)^{\lambda+1}}, \quad \lambda > -1,
\]

where \( * \) denotes convolution (Hadamard product). Let \( f \) be the normalized analytic function in the open unit disk \( U \). We define a new operator

\[
D_{\mu,m}^\lambda f(z) = [zF(z, m, c)]^\lambda \ast f(z).
\]

Moreover, we obtain some differential subordination and superordination results involving this operator. These results are obtained by investigating classes of admissible functions. Sandwich-type result is also studied.

Keywords: Hurwitz–Lerch zeta function, Hadamard product, Multiplier Transformation, Differential Subordination and Superordination

AMS Mathematics Subject Classification: 30C45
1 Introduction and Definitions

Let $H(U)$ denote the class of holomorphic functions in the open unit disk $U = \{z : z \in \mathbb{C} \mid |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$H[a, n] = \{f \in H(U), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U\}$$

and

$$A = \{f \in H(U), f(z) = z + a_z z^2 + \ldots, z \in U\}.$$

For $f_j \in A$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, \quad (j = 1, 2),$$

the Hadamard product (or convolution) $f_1 \ast f_2$ of $f_1$ and $f_2$ is defined by

$$(f_1 \ast f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k.$$

Let $F$ and $G$ be analytic functions in the unit disk $U$. The function $F$ is subordinate to $G$, written $F \prec G$ if $G$ is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In general, given two functions $F$ and $G$ which are analytic in $U$, the function $F$ is said to be subordinate to $G$, if there exist a function $w$ analytic in $U$ with

$$w(0) = 0 \text{ and } (\forall z \in U) : \left| w(z) \right| < 1$$

such that

$$(\forall z \in U) : F(z) = G(w(z)).$$

Now let us consider the generalized Hurwitz–Lerch zeta function

$$\phi_{\mu}(z, m, c) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+c)^m},$$

$$(1.1)$$

$$z \in \mathbb{C}, |z| < 1, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \mu, m \in \mathbb{C},$$

introduced by Goyal and Laddha [11]. Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k = k!$) and $(\lambda)_k$ given in terms of the Gamma functions can be written as

$$(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)} = x(x + 1)(x + k - 1) \quad \text{for } k = 1, 2, 3, \ldots, \quad x \in \mathbb{C} \quad (x)_0 = 1.$$
Note that, the families and special cases of the Hurwitz–Lerch zeta function are studied by many authors (among them) Shy-Der Lin & Srivastava [11] and Kanemitsu et.al. [10].

Now we define the function $F(z,m,c)$ given by

$$F(z,m,c) = (1+c)^m \phi_\mu(z,s,c) = \sum_{k=0}^{\infty} \left( \frac{1+c}{k+c} \right)^m \frac{(\mu)_k}{k!} z^k,$$

and

$$zF(z,m,c) = z \left( 1+c \right)^m \phi_\mu(z,m,c) = \sum_{k=0}^{\infty} \left( \frac{1+c}{k+c} \right)^m \frac{(\mu)_{k-1}}{(k-1)!} z^{k+1}.$$

Thus

$$zF(z,m,c) = \sum_{k=1}^{\infty} \left( \frac{1+c}{k+c} \right)^m \frac{(\mu)_{k-1}}{(k-1)!} z^k,$$

for $z \in \mathbb{C}$, $|z|<1$, $c \in \mathbb{C}$, $\mu,m \in \mathbb{C}$.

Now we introduce the function $[zF(z,m,c)]^{-1}$ as the following:

$$[zF(z,m,c)]*[zF(z,m,c)]^{-1} = \frac{z}{(1-z)^{\lambda+1}}, \lambda > -1, z \in \mathbb{C}, |z|<1,$$

for $c \in \mathbb{C}$, $\mu,m \in \mathbb{C}$.

Corresponding to the function $[zF(z,m,c)]^{-1}$ we define a multiplier transformation $D_{\mu,c}^{z,m}$ on $A$ and by Hadamard product for function $f \in A$, we have

$$D_{\mu,c}^{z,m} = [zF(z,m,c)]^{-1} \ast f(z). \quad (1.2)$$

Since

$$[zF(z,m,c)]^{-1} = \sum_{k=1}^{\infty} \left( \frac{k+c}{1+c} \right)^m \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} z^{-k},$$
Therefore we have

\[
D_{\mu,c}^{\lambda,m} f(z) = z + \sum_{k=2}^{\infty} \left( \frac{k + c}{1 + c} \right)^m \frac{(\lambda + 1)_{k-1}}{(\mu)_{k-1}} a_k z^k .
\]  

(1.3)

In view of (1.3) we obtain

\[
z (D_{\mu,c}^{\lambda,m} f(z))' = (\lambda + 1)D_{\mu,c}^{\lambda+1,m} f(z) - \lambda D_{\mu,c}^{\lambda,m} f(z) ,
\]

(1.4)

\[
z (D_{\mu+1,c}^{\lambda,m} f(z))' = \mu D_{\mu+1,c}^{\lambda,m} f(z) - (\mu - 1)D_{\mu+1,c}^{\lambda-1,m} f(z)
\]

(1.5)

and also

\[
z (D_{\mu,c}^{\lambda,m} f(z))' = (c + 1)D_{\mu,c}^{\lambda,m+1} f(z) - cD_{\mu,c}^{\lambda,m} f(z).
\]

(1.6)

It is clear that \( D_{\mu,c}^{\lambda,m} \) are multiplier transformations. For \( m \in \mathbb{Z} \), \( c \geq 1, \mu = 1 \) and \( \lambda = 0 \) the operator \( D_{\mu,c}^{\lambda,m} \) were studied by Cho and Srivastava [3]. For \( m \in \mathbb{Z} \), \( c = 1, \mu = 1 \) and \( \lambda = 0 \) the operator \( D_{\mu,c}^{\lambda,m} \) were studied by Uralegaddi and Somanatha [2], for \( m = -1, \mu = 1 \) and \( \lambda = 0 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the integral operator studied by Owa and Srivastava [12], for any negative real number \( m \) and \( \mu = 1, c = 1, \lambda = 0 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the integral operator studied by Jung et. al. [6], for any non-negative integer number \( m \) and \( \mu = 1, c = 0, \lambda = 0 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the differential operator defined by Salagean [5], for \( m = 0, \mu = 1, \lambda > 1 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the differential operator defined by Ruscheweyh [14], for \( m \in \mathbb{Z} \), \( \lambda = 0, \mu = 1 \) the operator \( D_{\mu,c}^{\lambda,m} \) are closely related to the multiplier transformations studied by Flett [17], for \( \mu = 1 \) and \( \lambda > -1 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the multiplier transformations defined by Al-Shaqsi and Darus [9], for \( c = 0, \mu = 1 \) and \( \lambda > 1 \) the operator \( D_{\mu,c}^{\lambda,m} \) is the derivative operator given by Al-Shaqsi and Darus [8], for \( c = 0, m, \lambda \in \mathbb{N}_0 \) and \( \mu \in \mathbb{N} \) the operator \( D_{\mu,c}^{\lambda,m} \) is the linear operator defined by the authors [1]. In particular, we note that \( D_{1,c}^{0,0} = f(z) \) and \( D_{1,0}^{0,1} = zf'(z) \).
Let $p, h \in H(U)$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If
\[ p \text{ and } \psi(p(z),zp'(z),z^2p''(z);z) \]
are univalent and if $p$ satisfies the (second-order) differential superordination
\[ h(z) \prec \psi(p(z),zp'(z),z^2p''(z);z), \quad z \in U \]  
then $p$ is called a solution of the differential superordination of (1.7).

An analytic function $q$ is called a subordinant of the differential superordination, if $q \prec p$ for all $p$ satisfying (1.7). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (1.7) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of $U$). On the other hand, an analytic function $q$ is said to be dominant if $p \prec q$ for all $p$ satisfying
\[ \psi(p(z),zp'(z),z^2p''(z);z) \prec h(z), \quad z \in U \]  
then $q$ is called a subordinant of the differential superordination of (1.7).

A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.8) is said to be the best dominant. Recently Miller and Mocanu [16] obtained conditions on $h,q$ and $\psi$ for which the following implication holds:
\[ h(z) \prec \psi(p(z),zp'(z),z^2p''(z);z) \Rightarrow q(z) \prec p(z) \quad (z \in U). \]

Denoted by $Q$ the set of all functions $q$ that are analytic and injective on $\overline{U} \setminus E(q)$ where
\[ E(q) = \{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \} \]
and are such that $q'(z) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$ and $Q(1) = Q_1$.

**Definition 1.1.** [15, Definition 2.3a, p. 27]. Let $\Omega$ be a set in $\mathbb{C}$, $q \in Q$ and $n$ be a positive integer. The class of admissible functions $\Psi_i[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega$ whenever $r = q(\zeta), s = k \zeta q'(\zeta)$, and
\[ \Re \left\{ \frac{L+1}{s} \right\} \geq k \Re \left\{ \frac{\zeta q^n(\zeta)}{q'(\zeta)} + 1 \right\}, \quad (z \in U, \zeta \in \partial U \setminus E(q), k \geq n). \]

We write $\Psi_i[\Omega,q]$ as $\Psi[\Omega,q]$. 

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Definition 1.2. [16, Definition 3, p. 817] Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in \mathcal{H}[a,n] \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Psi'_n[\Omega,q] \) consists of those functions \( \psi : \mathbb{C}^1 \times \overline{U} \to \mathbb{C} \) that satisfy the admissibility condition \( \psi(r,s,t;\zeta) \in \Omega \) whenever \( r = q(z), s = zq'(z)/m \), and

\[
\Re \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},
\]

\( z \in U, \zeta \in \partial U \) and \( m \geq n \geq 1 \). In particular we write \( \Psi'_n[\Omega,q] \) as \( \Psi'[\Omega,q] \).

Theorem 1.1. [15, Theorem 2.3b, p. 28]. Let \( \psi \in \Psi'_n[\Omega,q] \) with \( q(0) = a \). If \( p \in \mathcal{H}[a,n] \) satisfies

\[
\psi(p(z),zp'(z),z^2p''(z);z) \in \Omega,
\]

then \( p(z) \prec q(z) \).

If the behavior of \( q \) is not known on the boundary of \( U \), Miller and Mocanu [15] introduced the following limiting procedure to prove that \( p \prec q \).

Corollary 1.1. [15, Corollary 2.3b.1, p. 30]. Let \( \Omega \subset \mathbb{C} \) and \( q \) be univalent in \( U \), with \( q(0) = a \). Let \( \psi \in \Psi'_n[\Omega,q\rho] \) for some \( \rho \in (0,1) \), where \( q_\rho(z) = q(\rho z) \).

If \( p \in \mathcal{H}[a,n] \) and \( \psi(p(z),zp'(z),z^2p''(z);z) \in \Omega \), then \( p(z) \prec q(z) \).

Theorem 1.2. [16, Theorem 1, p. 818]. Let \( \psi \in \Psi'_n[\Omega,q] \) with \( q(0) = a \). If \( p \in \mathcal{Q}(a) \) and \( \psi(p(z),zp'(z),z^2p''(z);z) \) is univalent in \( U \), then

\[
\Omega \subset \{ \psi(p(z),zp'(z),z^2p''(z);z) : z \in U \}
\]

Then \( q(z) \prec p(z) \).

In the present paper, we shall use the method of differential subordination and Superordination introduced by Miller and Mocanu [15, Theorem 2.3b, p. 28] and [16, Theorem 1, p. 818] to derive certain properties of multiplier transformation \( D_{\lambda,\mu}^{m,c} \) and sandwich-type result is obtained.

2 Subordination Results

First, the following class of admissible functions is required in our first result.

Definition 2.1. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q(z) \in \mathcal{Q}_1 \cap \mathcal{H}[q(0),1] \). The class of admissible functions \( \Pi_n[\Omega,q] \) consists of those functions \( \pi : \mathbb{C}^1 \times U \to \mathbb{C} \) that satisfy the admissibility condition
\[ \pi(u,v,w;z) \notin \Omega \]

whenever

\[ u = q(\zeta), \quad v = \frac{k \zeta q'(\zeta) + \mu q(\zeta)}{\mu} \]

\[ \Re \left\{ \frac{(\mu - 1)(v - u)}{v - u} - (2 \mu - 1) \right\} \geq k \Re \left\{ \frac{\zeta q''(z)}{q'(z)} + 1 \right\}, \]

\[ (z \in U, \zeta \in \partial U \setminus E(q), k \geq 1). \]

Now we will derive our first result.

**Theorem 2.1.** Let \( \pi \in \Pi_{n}[\Omega,q] \). If \( f \in A \) satisfies

\[ \{ \pi((D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))', (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))''), (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))'; z) : z \in U \} \subset \Omega \]  \hspace{1cm} (2.1)

Then

\[ (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))' \prec q(z). \]

**Proof.** Define the analytic function \( p \) in \( U \) by

\[ p(z) = (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))'. \]  \hspace{1cm} (2.2)

In view of the relation (1.5) and from (2.2) we get

\[ (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))' = \frac{zp'(z) + \mu p(z)}{\mu}. \]  \hspace{1cm} (2.3)

Further, a simple computation shows that

\[ (D_{\mu^{1,1,\mu}^{1,1,\mu}}^{1,1,\mu}f(z))' = \frac{z^2 p''(z) + 2 \mu z p'(z) + \mu (\mu - 1)p(z)}{\mu (\mu - 1)}. \]  \hspace{1cm} (2.4)
Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$
\begin{align*}
    u(r,s,t) &= r; \\
v(r,s,t) &= \frac{s + \mu r}{\mu} \\
w(r,s,t) &= \frac{t + 2\mu s + \mu(\mu-1)r}{\mu(\mu-1)}.
\end{align*}
$$

(2.5)

Let

$$
\psi(r,s,t;z) = \pi(u,v,w;z)
$$

$$
= \pi(r, \frac{s + \mu r}{\mu}, \frac{t + 2\mu s + \mu(\mu-1)r}{\mu(\mu-1)};z)
$$

(2.6)

By making use of Theorem 1.1, and using equations (2.2), (2.3) and (2.4), also from (2.6), we obtain

$$
\psi(p(z),zp'(z),z^2p''(z);z) = \pi((D_{\mu+1}^{\lambda,m}f(z))',(D_{\mu}^{\lambda,m}f(z))',(D_{\mu}^{\lambda,m-1}f(z))';z)'
$$

(2.7)

Hence (2.1) becomes

$$
\pi((D_{\mu+1}^{\lambda,m}f(z))',(D_{\mu}^{\lambda,m}f(z))',(D_{\mu}^{\lambda,m-1}f(z))';z)
$$

$$
= \psi(p(z),zp'(z),z^2p''(z);z) \in \Omega.
$$

(2.8)

It remains to show that the admissibility condition for $\pi \in \Pi_n[\Omega,q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.1.

Note that

$$
\frac{t}{s} + 1 = \frac{(\mu-1)(w-u)}{v-u} - (2\mu-1),
$$

and hence $\psi \in \Psi_n[\Omega,q]$. By Theorem 1.1, $p(z) \prec q(z)$, or $(D_{\mu+1}^{\lambda,m}f(z))' \prec q(z)$.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where $h$ is a conformal mapping of $U$ onto $\Omega$. In this case the class $\Pi_n[h(U),q]$ is written as $\Pi_n[h,q]$.

The following result is an immediate consequence of Theorem 2.1.
Theorem 2.2. Let $\pi \in \Pi_n[h,q]$ with $q(0)=1$. If $f(z) \in A$ satisfies

$$\pi((D_{\lambda,m}^n f(z))', (D_{\lambda,m}^n f(z))', (D_{\lambda,m}^n f(z))', z) \prec h(z),$$

Then

$$(D_{\lambda,m}^n f(z))' \prec q(z).$$

By making use Corollary 1.1, we give an extension of Theorem 2.1 in the case where the behavior of $q$ on $\partial U$ is not known.

Corollary 2.1. Let $\Omega \subseteq \mathbb{C}$ and let $q$ be univalent in $U$, $q(0)=1$. Let $\pi \in \Pi_n[\Omega, \rho \pi]$ for some $\rho \in (0,1)$ where $q_{\rho}(z) = q(\rho z)$. If $f \in A$ and

$$\pi((D_{\lambda,m}^n f(z))', (D_{\lambda,m}^n f(z))', (D_{\lambda,m}^n f(z))', z) \in \Omega,$$

then

$$(D_{\lambda,m}^n f(z))' \prec q(z).$$

Proof. Theorem 2.1 yields $(D_{\lambda,m}^n f(z))' \prec q_{\rho}(z)$. The result is now deduced from $q_{\rho}(z) \prec q(z)$.

Theorem 2.3. Let $h$ and $q$ be univalent function in $U$, with $q(0)=1$ and set $q_{\rho}(z) = q(\rho z)$. and $h_{\rho}(z) = h(\rho z)$. Let $\pi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$, satisfy one of the following conditions:

(i) $\pi \in \Pi_n[h,q_{\rho}]$ for some $\rho \in (0,1)$, or

(ii) there exists $\rho_0 \in (0,1)$, such that $\pi \in \Pi_n[h_{\rho},q_{\rho}]$ for all $\rho \in (\rho_0,1)$.

If $f \in A$ satisfies (2.9), then $(D_{\lambda,m}^n f(z))' \prec q(z)$.

Proof. Following the same argument in [15, Theorem 2.3d, p. 30], we have

(i) By applying Theorem 2.1 we obtain $(D_{\lambda,m}^n f(z))' \prec q_{\rho}(z)$. Since $q_{\rho}(z) \prec q(z)$, we deduce $(D_{\lambda,m}^n f(z))' \prec q(z)$.

(ii) If we let $(D_{\lambda,m}^n f(z))' = (D_{\lambda,m}^n f(\rho z))'$, then
\[ \pi((D_{\mu+\lambda} f)(z)')' (D_{\mu} f)(z)'; \rho z) \]
\[ = \pi((D_{\mu+\lambda} f (\rho z))', (D_{\mu} f (\rho z))', (D_{\mu-1} f (\rho z)'); \rho z) \in h_\rho (U). \]

By using Theorem 2.1 and the comment associated with (2.8) with \( w(z) = \rho z \) which mapping \( U \) into \( \overline{U} \), we obtain \( (D_{\mu+\lambda} f (\rho z))' \prec q_\rho (z) \), for \( \rho \in (\rho_0, 1) \). By letting \( \rho \to 1 \), we obtain \( (D_{\mu+\lambda} f (z))' \prec q(z) \).

The next Theorem yields best dominant of the differential subordination (2.9)

**Theorem 2.4.** Let \( h \) be univalent in \( U \), and \( \pi : \mathbb{C} \times U \to \mathbb{C} \). Suppose the differential equation

\[ \pi \left( \left. q(z), zq'(z) + \mu q(z) \right|, \frac{z^2 q''(z) + 2\mu z q'(z) + \mu (\mu - 1) q(z)}{\mu (\mu - 1)} \right) = h(z) \]  

(2.10)

has a solution \( q \) with \( q(0) = 1 \) and one of the following conditions is satisfied:
(i) \( q(z) \in Q \) and \( \pi \in \Pi_n[h,q] \),
(ii) \( q(z) \) is univalent in \( U \) and \( \pi \in \Pi_n[h,q,\rho] \), for some \( \rho \in (0, 1) \) or
(iii) \( q(z) \) is univalent in \( U \) and there exists \( \rho_0 \in (0, 1) \) such that \( \pi \in \Pi_n[h, q, \rho] \),

for all \( \rho \in (\rho_0, 1) \).

If \( f(z) \in A \) satisfies (2.9), then \( (D_{\mu+\lambda} f)(z) \) \( \prec q(z) \) and \( q(z) \) is the best dominant.

**Proof.** By using the same method given by [15, theorem 2.3e, p. 31], we deduce that from Theorems 2.1 and 2.2 above, \( q \) is a dominant of (2.9). Since \( q \) satisfies (2.10), it is a solution of (2.9) and therefore \( q \) will be dominated by all dominants of (2.9). Hence \( q \) will be the best dominant of (2.9).

### 3 Superordination and Sandwich Results

In this section the corresponding differential superordination problem is investigated and sandwich-type result is given.

**Definition 3.1.** Let \( \Omega \) be a set in \( \mathbb{C} \), \( q(z) \in H[q(0),1] \) with \( zq'(z) \neq 0 \). The class of admissible functions \( \Pi_n[q,\Omega] \) consists of those functions
\[ \pi : \mathbb{C} \times \overline{U} \to \mathbb{C} \]  

that satisfy the admissibility condition \( \pi(u, v, w; \zeta) \in \Omega \) whenever
Theorem 3.1. Let $\pi \in \Pi^{'n}[\Omega,q]$. If $f \in A, (D^\lambda_{\mu+1,\nu} f (z))' \in Q_1$ and

$$
\pi((D^\lambda_{\mu+1,\nu} f (z))',(D^\lambda_{\mu,\nu} f (z))', (D^\lambda_{\mu-1,\nu} f (z))'; z)
$$

is univalent in $U$, then

$$
\Omega \subset \{ \pi((D^\lambda_{\mu+1,\nu} f (z))',(D^\lambda_{\mu,\nu} f (z))', (D^\lambda_{\mu-1,\nu} f (z))'; z) : z \in U \}. \tag{3.1}
$$

implies $q(z) \prec (D^\lambda_{\mu+1,\nu} f (z))'$.

Proof. Let $p$ be defined by (2.2) and $\pi$ by (2.6). Since $\pi \in \Pi^{'n}[\Omega,q]$, (2.7) and (3.1) yield

$$
\Omega \subset \{ \pi(p(z),zp'(z),z^2p''(z); z) : z \in U \}.
$$

From (2.5), the admissibility condition for $\pi \in \Pi^{'n}[\Omega,q]$, is univalent to the admissibility condition for $\psi$ as given in Definition 3.1. Hence $\psi \in \Psi^{'n}[\Omega,q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or $q(z) \prec (D^\lambda_{\mu+1,\nu} f (z))'$.

Similarly as in the previous section, we next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where $h$ is a conformal mapping of $U$ onto $\Omega$. In this case the class $\Pi^{'n}[h(U),q]$ is written as $\Pi^{'n}[h,q]$.

The following result is an immediate consequence of Theorem 2.5.

Theorem 3.2. Let $q(z) \in H[q(0),1], h(z) \text{ be analytic in } U \text{ and } \pi \in \Pi^{'n}[h,q]$. If $f \in A, (D^\lambda_{\mu+1,\nu} f (z))' \in Q_1$ and

$$
\pi((D^\lambda_{\mu+1,\nu} f (z))',(D^\lambda_{\mu,\nu} f (z))', (D^\lambda_{\mu-1,\nu} f (z))'; z)
$$

is univalent in $U$, then

$$
\Omega \subset \{ \pi((D^\lambda_{\mu+1,\nu} f (z))',(D^\lambda_{\mu,\nu} f (z))', (D^\lambda_{\mu-1,\nu} f (z))'; z) : z \in U \}.
$$
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is univalent in $U$, then

$$h(z) \prec \pi((D_{\mu+\lambda \mu}^{1,m} f (z))', (D_{\mu+\lambda \mu}^{1,m} f (z))'), (D_{\mu-1,\mu}^{1,m} f (z))'; z) : z \in U$$

implies $q(z) \prec (D_{\mu+\lambda \mu}^{1,m} f (z))'$.

**Theorem 3.3.** Let $h$ be analytic in $U$, and $\pi: \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. Suppose the differential equation

$$\pi \left( q(z), \frac{zq'(z) + \mu q(z)}{\mu}, \frac{z^2q''(z) + 2\mu z q'(z) + \mu(\mu-1)q(z)}{\mu(\mu-1)}; z \right) = h(z)$$

has a solution $q \in \mathcal{Q}_1$. If $\pi \in \Pi_1[\hbar, q, f] \cap \Pi_1[D_{\mu+1,\mu}^{1,m} f (z)]', \in \mathcal{Q}_1$ and

$$\pi((D_{\mu+1,\mu}^{1,m} f (z))', (D_{\mu+1,\mu}^{1,m} f (z))'), (D_{\mu-1,\mu}^{1,m} f (z))'; z) : z \in U$$

is univalent in $U$, then

$$h(z) \prec \pi((D_{\mu+1,\mu}^{1,m} f (z))', (D_{\mu+1,\mu}^{1,m} f (z))'), (D_{\mu-1,\mu}^{1,m} f (z))'; z) : z \in U$$

implies $q(z) \prec (D_{\mu+1,\mu}^{1,m} f (z))'$, and $q(z)$ is the best subordinant.

**Proof.** The proof is similar to the proof of Theorem 2.4 and is omitted.

Combining Theorems 2.2 and 3.3, we obtain the following sandwich-type theorem.

**Corollary 3.1.** Let $h_1(z)$ and $q_1(z)$ be analytic functions in $U$, $h_2(z)$ be univalent in $U$, $q_1(z), q_2(z) \in \mathcal{Q}_1$, with $q_1(0) = q_2(0) = 1$, and $\pi \in \Pi_1[h_2, q_1] \cap \Pi_1[\hbar, q_1]$. If $f \in A \cap \Pi_1[D_{\mu+1,\mu}^{1,m} f (z)]', \in H[q(0), 1] \cap \mathcal{Q}_1$ and

$$\pi((D_{\mu+1,\mu}^{1,m} f (z))', (D_{\mu+1,\mu}^{1,m} f (z))'), (D_{\mu-1,\mu}^{1,m} f (z))'; z)$$

is univalent in $U$, then

$$h_1(z) \prec \pi((D_{\mu+1,\mu}^{1,m} f (z))', (D_{\mu+1,\mu}^{1,m} f (z))'), (D_{\mu-1,\mu}^{1,m} f (z))'; z) \prec h_2(z)$$

implies $q_1(z) \prec (D_{\mu+1,\mu}^{1,m} f (z))' \prec q_2(z)$.

**4 Open Problem**

The definitions, theorems and corollaries we established can be extended by using the concept of the strong differential subordination introduced in [7] by Antonino and Romaguera and studied in [4] by Oros and Oros.
ACKNOWLEDGMENT: The work here is fully supported by UKM-GUP-TMK-07-02-107, Malaysia.

References


