

## Research Article

# Homotopy Perturbation Method for Solving Reaction-Diffusion Equations

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The homotopy perturbation method is applied to solve reaction-diffusion equations. In this method, the trial function (initial solution) is chosen with some unknown parameters, which are identified using the method of weighted residuals. Some examples are given. The obtained results are compared with the exact solutions, revealing that this method is very efficient and the obtained solutions are of high accuracy.

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## 1. Introduction

In this paper, we consider a reaction-diffusion process governed by the nonlinear ordinary differential equation [1]:

$$y''(x) + y^n(x) = 0, \quad 0 < x < L, \quad (1.1)$$

with boundary conditions

$$y(0) = y(L) = 0, \quad (1.2)$$

where  $y(x)$  represents the steady-state temperature for the corresponding reaction-diffusion equation with the reaction term  $y^n$ ;  $n$  is the power of the reaction term (heat source), generally it follows  $n > 0$ ,  $L$  is the length of the sample (heat conductor). The physical interpretation of (1.1) was given in [1].

Recently, various different analytical methods were applied to nonlinear equations arising in engineering applications, such as the homotopy perturbation method [2–10], and exp-function method [11, 12], a complete review is available in [13]. This problem was studied

by Lesnic using Adomian method [1], and by Mo [14] using variational method. In this paper, the homotopy perturbation method [2, 3, 13] is applied to the discussed problem, and the obtained results show that the method is very effective and simple.

## 2. Homotopy perturbation method

In order to use the homotopy perturbation, we construct a homotopy in the form [2, 3, 13]

$$(1 - p)(y'' - y_0'') + p(y'' + y^n) = 0 \quad (2.1)$$

with initial approximation

$$y_0(x) = ax(1 - x) = ax - ax^2, \quad (2.2)$$

where  $a$  is an unknown constant to be further determined. It is obvious that (2.2) satisfies the boundary conditions.

We rewrite (2.1) in the form of

$$y'' + 2a - p(2a - y^n) = 0. \quad (2.3)$$

We suppose the solution of (2.3) has the form

$$y(x) = y_0(x) + py_1(x) + p^2y_2(x) + \dots \quad (2.4)$$

Substituting (2.4) into (1.1) and equating the terms with the identical powers of  $p$ , we can solve  $y_0, y_1, y_2, \dots$  sequentially with ease. Setting  $p = 1$ , we obtain the approximate solution of (1.1) in the form of

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \dots \quad (2.5)$$

To illustrate its solution procedure, we consider some special cases.

*Case 1* ( $n = 2$ ). Under such case, we can easily obtain sequentially

$$\begin{aligned} y_0'' &= -2a, \\ y_1'' &= 2a - y_0^2, \\ y_2'' &= -2y_0y_1. \end{aligned} \quad (2.6)$$

We, therefore, obtain the approximate solution in the form of

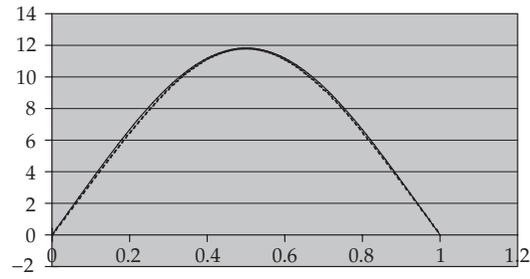
$$y(x) = ax(1 - x) + ax^2 - a^2 \left( \frac{1}{30}x^6 - \frac{1}{10}x^5 + \frac{1}{12}x^4 \right) - \left( a - \frac{1}{60}a^2 \right)x. \quad (2.7)$$

In order to identify the unknown constant  $a$  in (2.7), we apply the method of weighted residuals. Substituting (2.7) into (1.1) results in the following residual:

$$R(x, a) = y''(x) + y^n(x). \quad (2.8)$$

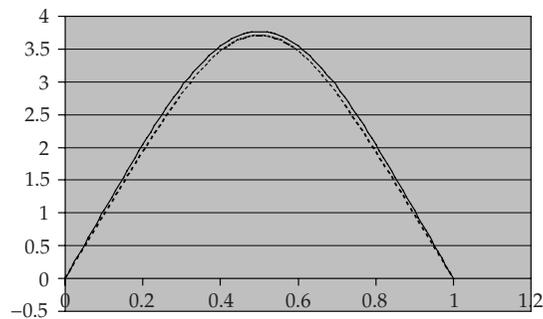
It is obvious that  $R(0, a) = 0$  and  $R(1, a) = 0$ . We locate at  $x = 1/3$ , and set  $R(1/3, a) = 0$ , yielding the result

$$a = 45.4205. \quad (2.9)$$



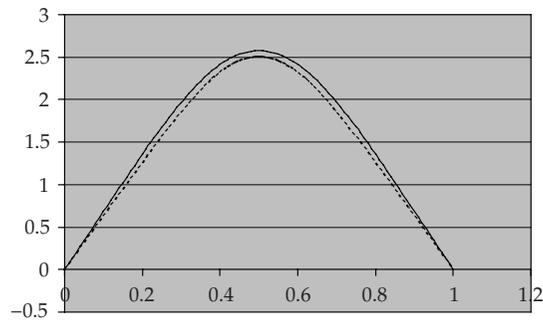
----- Exact solution  
— Approximate solution

(a)  $n = 2$



----- Exact solution  
— Approximate solution

(b)  $n = 3$



----- Exact solution  
— Approximate solution

(c)  $n = 4$

**Figure 1:** Comparison of approximate solutions with exact ones. Continued line: approximate solution; discontinued line: exact solution.

Case 2 ( $n = 3$ ). The solution procedure is the same as that for Case 1. We can easily obtain the following linear equations:

$$\begin{aligned} a &= 14.2657, \\ y_1'' &= 2a - y_0^3, \\ y_2'' &= -3y_0^2 y_1. \end{aligned} \quad (2.10)$$

We obtain the following second-order approximate solution:

$$y(x) = ax(1-x) + ax^2 - a^3 \left( -\frac{1}{56}x^8 + \frac{1}{14}x^7 - \frac{1}{10}x^6 + \frac{1}{20}x^5 \right) - \left( a - \frac{1}{280}a^3 \right) x. \quad (2.11)$$

Similarly, we locate at  $x = 1/3$ , and set  $R(1/3, a) = 0$  to identify the unknown constant, which reads  $a = 14.2657$ .

Case 3 ( $n = 4$ ). By the same manipulation as illustrated in above cases, we obtain

$$\begin{aligned} y_0'' &= -2a, \\ y_1'' &= 2a - y_0^4, \\ y_2'' &= -4y_0^3 y_1, \\ y(x) &= ax(1-x) + ax^2 - a^4 \left( \frac{1}{90}x^{10} - \frac{1}{18}x^9 + \frac{3}{28}x^8 - \frac{2}{21}x^7 + \frac{1}{30}x^6 \right) - (a - 7.9410^{-4}a^4)x. \end{aligned} \quad (2.12)$$

Using the method of weighted residuals, we set  $R(1/3a) = 0$ , resulting in  $a = 9.6320$ .

Figure 1 shows the remarkable accuracy of the obtained results.

### 3. Conclusion

The homotopy perturbation method deforms a complex problem under study to a simple problem routinely. If initial guess is suitably chosen, one iteration is enough, making the method a most attractive one. The method is of remarkable simplicity, while the obtained results are of utter accuracy on the whole solution domain. The method can be applied to various other nonlinear problems without any difficulty.

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