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Research Article

Some Geometric Inequalities in a New Banach Sequence Space

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The difference sequence space $m(\phi, p, \Delta^{(r)})$, which is a generalization of the space $m(\phi)$ introduced and studied by Sargent (1960), was defined by Çolak and Et (2005). In this paper we establish some geometric inequalities for this space.

1. Introduction and preliminaries

Let $\mathscr{C}$ denote the space whose elements are finite sets of distinct positive integers. Given an element $\sigma \in \mathscr{C}$, we write $c(\sigma)$ for the sequence $(c_n(\sigma))$ such that $c_n(\sigma) = 1$ for $n \in \sigma$, and $c_n(\sigma) = 0$, otherwise. Further

$$\mathscr{C}_s = \left\{\sigma \in \mathscr{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\}, \tag{1.1}$$

that is, $\mathscr{C}_s$ is the set of those σ whose support has cardinality at most s, where s is a natural number.

Let w be the set of all real sequences and

$$\Phi = \left\{\phi = (\phi_n) \in w : \phi_1 > 0, \nabla\phi_k \geq 0,\ \nabla\left(\frac{\phi_k}{k}\right) \leq 0\,(k = 1,2,\dots)\right\}, \tag{1.2}$$

where $\nabla\phi_k = \phi_k - \phi_{k-1}$. For $\phi \in \Phi$, Sargent [1] introduced the following sequence space:

$$m(\phi) = \left\{x = (x_n) \in w : \sup_{s\geq 1} \sup_{\sigma\in\mathscr{C}_s} \left(\frac{1}{\phi_s}\sum_{n\in\sigma} |x_n|\right) < \infty\right\}. \tag{1.3}$$

In [2], the space $m(\phi)$ has been considered for matrix transformations and in [3] some of its geometric properties have been considered. Tripathy and Sen [4] extended $m(\phi)$ to $m(\phi,p), 1 \le p < \infty$. Recently, Çolak and Et [5] defined the space $m(\phi,p,\Delta^{(r)})$ by using the idea of difference sequences (see [6–8]).

Let r be a positive integer throughout. The operators $\Delta^{(r)},\Sigma^{(r)} : w \to w$ are defined by

$$\begin{gathered}(\Delta^{(1)}x)_k = (\Delta x)_k = x_k - x_{k+1},\\ (\Sigma^{(1)}x)_k = (\Sigma x)_k = \sum_{j=k}^{\infty} x_j \quad (k = 1,2,\dots),\\ \Delta^{(r)} = \Delta^{(1)} \circ \Delta^{(r-1)}, \quad \Sigma^{(r)} = \Sigma^{(1)} \circ \Sigma^{(r-1)}, \quad (r \ge 2),\\ \Sigma^{(r)} \circ \Delta^{(r)} = \Delta^{(r)} \circ \Sigma^{(r)} = id, \quad \text{the identity on } w.\end{gathered} \tag{1.4}$$

For $0 \le p < \infty$, the space $m(\phi,p,\Delta^{(r)})$ is defined as follows:

$$m(\phi,p,\Delta^{(r)}) = \left\{x \in w : \sup_{s\ge 1,\sigma\in\mathcal{C}_s} \left(\frac{1}{\phi_s}\sum_{n\in\sigma} |\Delta^{(r)}x_n|^p\right) < \infty\right\}, \tag{1.5}$$

which is a Banach space $(1 \le p < \infty)$ with the norm

$$\|x\|_{m(\phi,p,\Delta^{(r)})} = \sum_{i=1}^{r} |x_i| + \sup_{s\ge 1,\sigma\in\mathcal{C}_s} \frac{1}{\phi_s}\left(\sum_{n\in\sigma} |\Delta^{(r)}x_n|^p\right)^{1/p}, \tag{1.6}$$

and a complete p-normed space $(0 < p < 1)$ with the p-norm

$$\|x\|_{m_p(\phi,\Delta^{(r)})} = \sum_{i=1}^{r} |x_i|^p + \sup_{s\ge 1,\sigma\in\mathcal{C}_s} \frac{1}{\phi_s}\sum_{n\in\sigma} |\Delta^{(r)}x_n|^p. \tag{1.7}$$

In this paper, we will consider the case $1 < p < \infty$ to study some geometric properties of $m(\phi,p,\Delta^{(r)})$. We will examine the Banach-Saks property of type p, strict convexity and uniform convexity. The space $m(\phi,p), 1 \le p < \infty$ was defined by Tripathy and Sen [4] which is in fact $m(\phi,p,\Delta)$ with Δ replaced by *id*.

Let $1 < p < \infty$. A Banach space X is said to have the *Banach-Saks property of type p* or *property* $(BS)_p$ if every weakly null-sequence (x_k) has a subsequence (x_{k_i}) such that for some $C > 0$, the inequality

$$\left\|\sum_{i=0}^{n} x_{k_i}\right\|_X \le c(n+1)^{1/p}, \quad n = 1,2,3,\dots, \tag{1.8}$$

holds.

The property $(BS)_p$ for a Cesàro sequence space was considered in [9].

We find uniform convexity and strict convexity of our space through the Gurarii's modulus of convexity (see [10, 11]).

For a normed linear space X, the modulus of convexity defined by

$$\beta_X(\varepsilon) = \inf\left\{1 - \inf_{0\le\alpha\le 1} \|\alpha x + (1-\alpha)y\| : x,y \in S(X), \|x-y\| = \varepsilon\right\}, \tag{1.9}$$

is called the Gurarii's modulus of convexity, where $S(X)$ denotes the unit sphere in X and $0 < \varepsilon \leq 2$. If $0 < \beta_X(\varepsilon) < 1$, then X is uniformly convex and if $\beta_X(\varepsilon) \leq 1$, then X is strictly convex.

2. Main results

Theorem 2.1. *The space $m(\phi, p, \Delta^{(r)})$ has the Banach-Saks property of type p.*

Proof. We will prove the case $r = 1$ and the general case can be followed on the same lines. □

Let (ε_n) be a sequence of positive numbers for which $\sum_{n=1}^{\infty}\varepsilon_n \leq 1/2$. Let (x_n) be a weakly null sequence in $B(m(\phi, p, \Delta))$, the unit ball in $m(\phi, p, \Delta)$. Set $x_0 = 0$ and $z_1 = x_{n_1} = \Delta x_1$. Then there exists $s_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i \in \tau_1} z_1(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_1, \tag{2.1}$$

where τ_1 consists of the elements of σ which exceed s_1. Since $x_n \xrightarrow{w} 0 \Rightarrow x_n \to 0$ coordinatewise, there is $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{s_1} x_n(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_1, \quad \text{when } n \geq n_2. \tag{2.2}$$

Set $z_2 = x_{n_2} = \Delta x_2$. Then there exists $s_2 > s_1$ such that

$$\left\| \sum_{i \in \tau_2} z_2(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_2, \tag{2.3}$$

where τ_2 consists of the elements of σ which exceed s_2. Again using the fact $x_n \to 0$ coordinatewise, there exists $n_3 > n_2$ such that

$$\left\| \sum_{i=1}^{s_2} x_n(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_2, \quad \text{when } n \geq n_3. \tag{2.4}$$

Continuing this process, we can find two increasing sequences (s_i) and (n_i) such that

$$\begin{gathered} \left\| \sum_{i=1}^{s_j} x_n(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_j, \quad \text{when } n \geq n_{j+1}, \\ \left\| \sum_{i \in \tau_j} z_j(i) e_i \right\|_{m(\phi,p,\Delta)} < \varepsilon_j, \end{gathered} \tag{2.5}$$

where $z_j = x_{n_j} = \Delta x_j$ and τ_j consists of the elements of σ which exceed s_j. Note that $z_j(i)$ is a term in the sequence with fixed j and running i.

Since $\varepsilon_{j-1}+\varepsilon_j<1$, we have

$$\left(\frac{1}{\phi_s}\sum_{n\in\sigma}|z_j(n)|\right)\leq(\varepsilon_{j-1}+\varepsilon_j)<1, \tag{2.6}$$

for all $j\in\mathbb{N}$ and $s\geq 1$. Hence

$$\begin{aligned}
\left\|\sum_{j=1}^{n}z_j\right\|_{m(\phi,p,\Delta)} &= \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{s_{j-1}}z_j(i)e_i+\sum_{i=s_{j-1}+1}^{s_j}z_j(i)e_i+\sum_{i\in\tau_j}z_j(i)e_i\right)\right\|_{m(\phi,p,\Delta)}\\
&\leq \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{s_{j-1}}z_j(i)e_i\right)\right\|_{m(\phi,p,\Delta)}+\left\|\sum_{j=1}^{n}\left(\sum_{i=s_{j-1}+1}^{s_j}z_j(i)e_i\right)\right\|_{m(\phi,p,\Delta)}\\
&\quad+\left\|\sum_{j=1}^{n}\left(\sum_{i\in\tau_j}z_j(i)e_i\right)\right\|_{m(\phi,p,\Delta)}\\
&\leq\sum_{j=1}^{n}\left\|\left(\sum_{i=s_{j-1}+1}^{s_j}z_j(i)e_i\right)\right\|_{m(\phi,p,\Delta)}+2\sum_{j=1}^{n}\varepsilon_j,\\
\sum_{j=1}^{n}\left\|\sum_{i=s_{j-1}+1}^{s_j}z_j(i)e_i\right\|_{m(\phi,p,\Delta)}^{p} &= \sum_{j=1}^{n}\sup_{s\geq 1}\sup_{\tau_{j-1}\in\mathscr{C}_s}\left(\frac{1}{\phi_s}\sum_{i\in\tau_{j-1}}|z_j(i)|^p\right)\\
&\leq\sum_{j=1}^{n}\sup_{s\geq 1}\sup_{\sigma\in\mathscr{C}_s}\left(\frac{1}{\phi_s}\sum_{i\in\sigma}|z_j(i)|^p\right)\leq n.
\end{aligned} \tag{2.7}$$

Therefore by (2.7)

$$\left\|\sum_{j=1}^{n}z_j\right\|_{m(\phi,p,\Delta)}\leq n^{1/p}+1\leq 2n^{1/p} \tag{2.8}$$

since $\sum_{j=1}^{n}\varepsilon_j\leq 1/2$.

Hence $m(\phi,p,\Delta)$ has the Banach-Saks property of type p.

Remark 2.2. The above result can also be extended to the case when $r\neq 1$ and so the proof should also work for a more general case with Δ replaced by a matrix operator (transformation).

THEOREM 2.3. *The Gurarii's modulus of convexity for the space $X=m(\phi,p,\Delta)$ is*

$$\beta_X(\varepsilon)\leq 1-\left[1-\left(\frac{\varepsilon}{2}\right)^p\right]^{1/p}, \tag{2.9}$$

where $0<\varepsilon\leq 2$.

Proof. Let $x \in m(\phi, p, \Delta)$. Then

$$\|x\|_{m(\phi,p,\Delta)} = \|\Delta x\|_{m(\phi,p)} = |x_1| + \sup_{s\geq 1,\sigma\in\mathscr{C}_s} \frac{1}{\phi_s}\left[\sum_{n\in\sigma}|\Delta x_n|^p\right]^{1/p}. \tag{2.10}$$

Let $0 < \varepsilon \leq 2$ and consider the sequences

$$\begin{aligned} u = (u_n) &= \left(\left(\sum\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)\right)^{1/p}, \sum\left(\frac{\varepsilon}{2}\right), 0, 0, \ldots\right), \\ v = (v_n) &= \left(\left(\sum\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)\right)^{1/p}, \sum\left(-\frac{\varepsilon}{2}\right), 0, 0, \ldots\right). \end{aligned} \tag{2.11}$$

Then $\|\Delta u\|_{m(\phi,p)} = \|u\|_{m(\phi,p,\Delta)} = 1$, $\|\Delta v\|_{m(\phi,p)} = \|v\|_{m(\phi,p,\Delta)} = 1$, that is, $u, v \in S(m(\phi,p,\Delta))$ and $\|\Delta u - \Delta v\|_{m(\phi,p)} = \|u - v\|_{m(\phi,p,\Delta)} = \varepsilon$.

For $0 \leq \alpha \leq 1$,

$$||\alpha u + (1-\alpha)v||^p_{m(\phi,p,\Delta)} = ||\alpha\Delta u + (1-\alpha)\Delta v||^p_{m(\phi,p)} = 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|\left(\frac{\varepsilon}{2}\right)^p. \tag{2.12}$$

Hence

$$\inf_{0\leq\alpha\leq 1} ||\alpha u + (1-\alpha)v||^p_{m(\phi,p,\Delta)} = 1 - \left(\frac{\varepsilon}{2}\right)^p. \tag{2.13}$$

Therefore, for $p \geq 1$

$$\beta_X(\varepsilon) \leq 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{1/p}. \tag{2.14}$$

This completes the proof of the theorem. □

Corollary 2.4. (i) *If* $\varepsilon = 2$, *then* $\beta_X(\varepsilon) \leq 1$ *and hence* $m(\phi, p, \Delta)$ *is strictly convex.*

(ii) *If* $0 < \varepsilon < 2$, *then* $0 < \beta_X(\varepsilon) < 1$ *and hence* $m(\phi, p, \Delta)$ *is uniformly convex.*

Remark 2.5. Note that these results are best possible for the time being, that is, they cannot be readily generalized to the general case because our results also hold for general matrix transformation.

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